Numerical Analysis – Eigenvalue and Eigenvector

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Eigenvalue problem

$$Ax = \lambda x$$

λ : eigenvalue

x : eigenvector

※ spectrum: a set of all eigenvalue

Eigenvalue

- **Eigenvalue** λ

$$(A-\lambda I)x=0$$

if det $(A-\lambda I)\neq 0$ ➔ x=0(trivial solution)

∴ To obtain a non-trivial solution,

det $(A-\lambda I)=0$

➔ $$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & a_{11} \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}=0$$

;Characteristic equation

Properties of Eigenvalue

1) Trace $A = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$

2) det $A = \prod_{i=1}^{n} \lambda_i$

3) If A is symmetric, then the eigenvectors are orthogonal: $x_i^T x_j = \begin{cases} 0, & i \neq j \\ G_{ii} & i = j \end{cases}$

4) Let the eigenvalues of $A = \lambda_1, \lambda_2, \cdots \lambda_n$

then, the eigenvalues of $(A - aI)$

$$= \lambda_1 - a, \lambda_2 - a, \cdots, \lambda_n - a,$$

Geometric Interpretation of Eigenvectors

- Transformation Ax

  $Ax = \lambda x$: The transformation of an eigenvector is mapped onto the same line of *x*.

- Symmetric matrix → orthogonal eigenvectors

- Relation to Singular Value

  if ***A*** is singular → 0 $\in$ {eigenvalues}

Eg. Calculating Eigenvectors(I)

- Exercise

1) $\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$; symmetric, non-singular matrix

(λ= -1, -6)

2) $\begin{bmatrix} -5 & 1 \\ -2 & -2 \end{bmatrix}$; non-symmetric, non-singular matrix

(λ= -3, -4)

Eg. Calculating Eigenvectors(II)

3) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$; symmetric, singular matrix

(λ= 5, 0)

4) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$; non-symmetric, singular matrix

(λ= 7, 0)

Discussion

- symmetric matrix

  => orthogonal eigenvectors

- singular matrix

  => $0 \in \{eigenvalue\}$

- Investigation into SVD

Similar Matrices

Two $n \times n$ matrices A and B are **similar** if a matrix S exists with $A = S^{-1}BS$. The important feature of similar matrices is that they have the same eigenvalues. The next result follows from observing that if $\lambda\mathbf{x} = A\mathbf{x} = S^{-1}BS\mathbf{x}$, then $BS\mathbf{x} = \lambda S\mathbf{x}$. Also, if $\mathbf{x} \neq \mathbf{0}$ and S is nonsingular, then $S\mathbf{x} \neq \mathbf{0}$, so $S\mathbf{x}$ is an eigenvector of B corresponding to its eigenvalue λ.

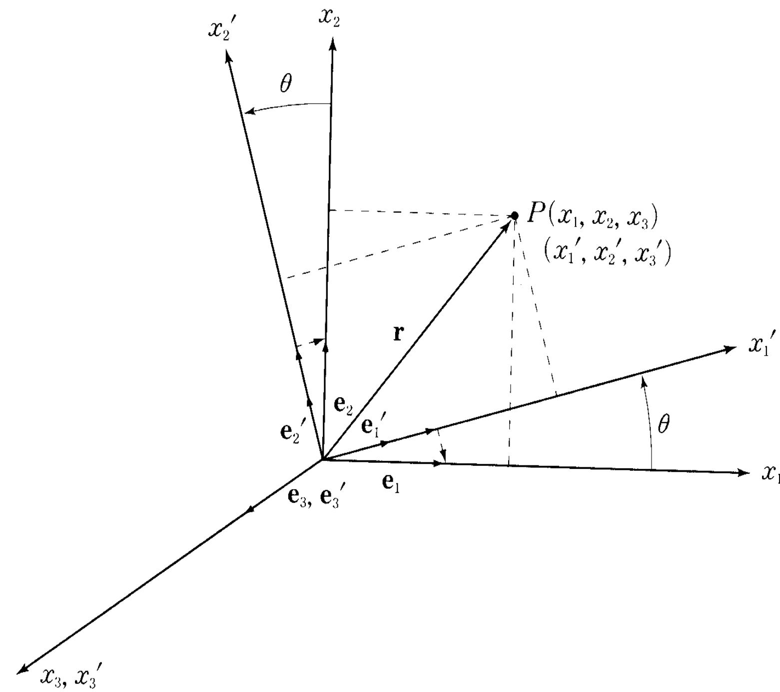
- Eigenvalues and eigenvectors of similar matrices

Suppose A and B are similar $n \times n$ matrices and λ is an eigenvalue of A with associated eigenvector $\mathbf{x}$. Then λ is also an eigenvalue of B. And, if $A = S^{-1}BS$, then $S\mathbf{x}$ is an eigenvector associated with λ and the matrix B.

Eg. Rotation matrix

Similarity Transformation

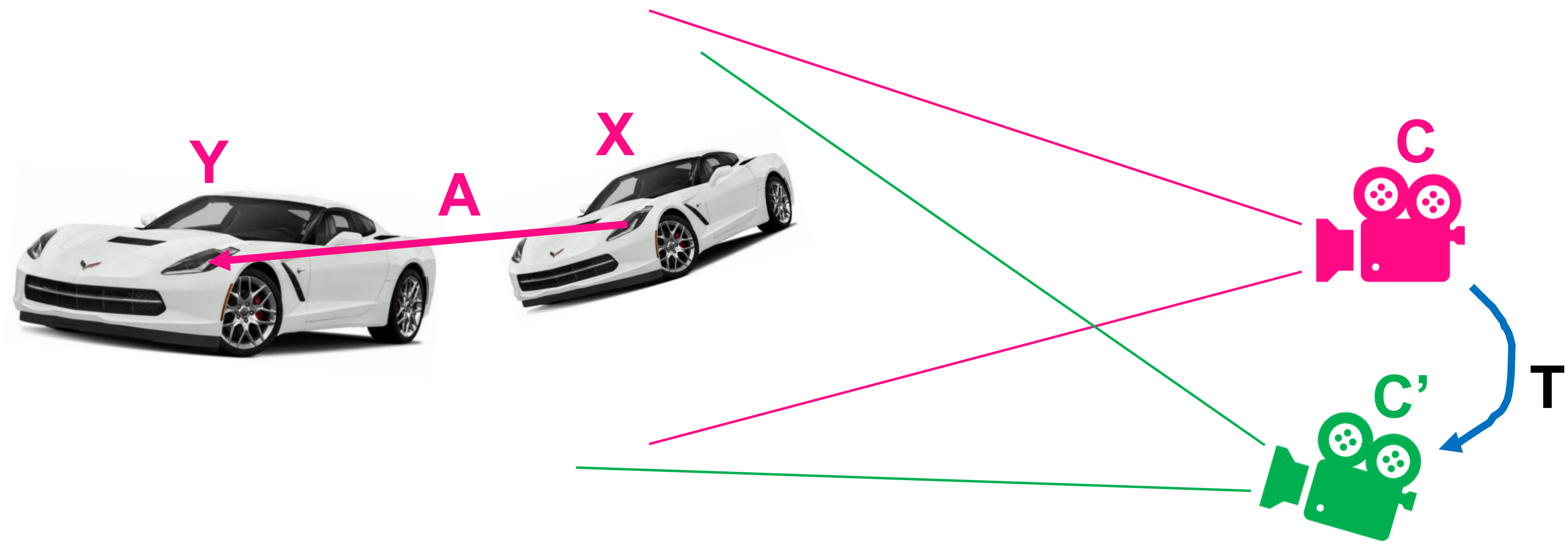
- Coordinate transformation
  - $x' = Rx$, $y' = Ry$



- Similarity transformation
  - $y = Ax$
  - $y' = Ry = RAx = RA(R^{-1} x') = RAR^{-1} x' = Bx'$

$$B = RAR^{-1}$$

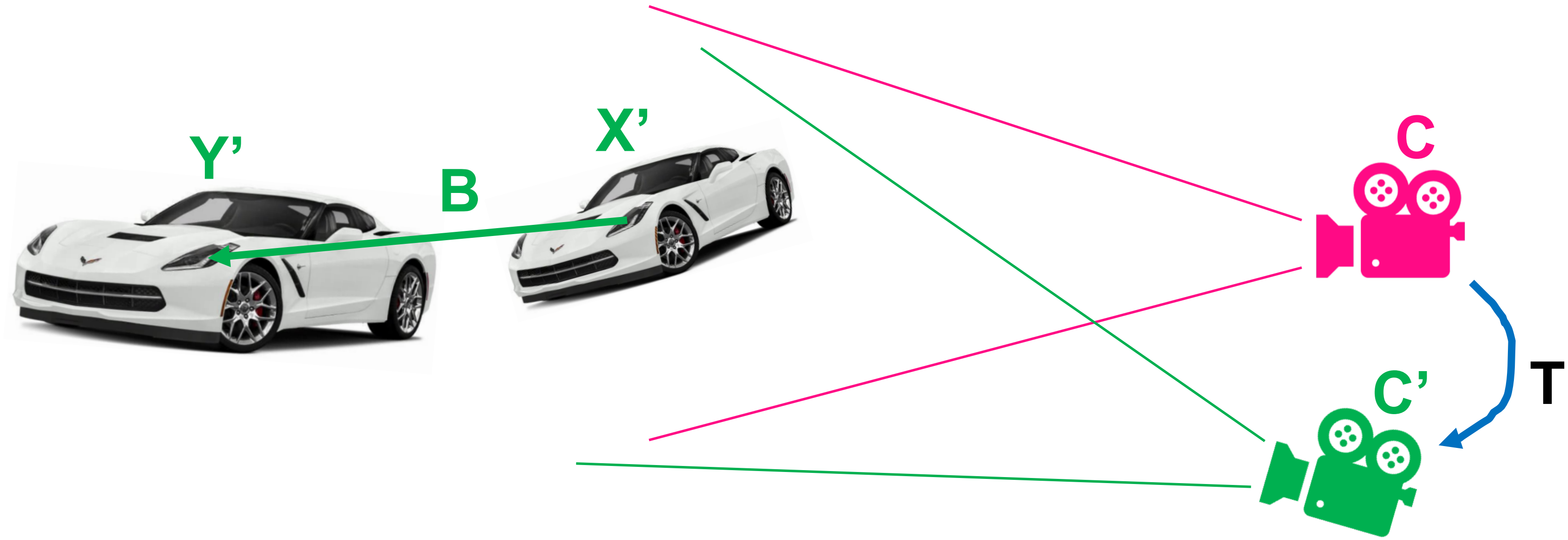
Similarity Transformation in CG



X'=TX (camera coordinate transformation)
Y=AX (object transformation seen from C)

What would be the object transformation B seen from C' ?

Y'=BX'



$X' = TX$ (camera coordinate transformation)

$Y = AX$ (object transformation seen from C)

What would be the object transformation B seen from C' ?

$$Y' = TY = TAX = TAT^{-1}X' = BX'$$

$$B = TAT^{-1}$$

Numerical Methods(I)

Power method

- Iteration formula

$$Ax^{(k)} = y^{(k+1)} = \lambda^{(k+1)} x^{(k+1)}$$

- for obtaining large λ

Eg. Power method

예제 3·17 다음 행렬에서 가장 큰 고유값과 해당 고유벡터를 멱승법으로 구하라.

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

풀이 초기값 $\mathbf{x}^{(0)}$을

$$\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

로 놓고, 식(3·57a)를 이용하여 계산하면,

$$\mathbf{A}\mathbf{x}^{(0)} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix} = \mathbf{y}^{(1)}$$

이것을 단위성분이 되도록 식(3·57b)와 같이 변형하자.

$$\begin{bmatrix} -3 \\ 0 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda^{(1)}\mathbf{x}^{(1)}$$

$\mathbf{x}=\mathbf{x}^{(1)}$을 식(3·57a)에 대입하여 계산하면 다음과 같다.

$$\mathbf{A}\mathbf{x}^{(1)} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

이 식을 식(3·57b)의 우변과 같이 변형하면,

$$\begin{bmatrix} -5 \\ 2 \end{bmatrix} = -5\begin{bmatrix} 1 \\ -\frac{2}{5} \end{bmatrix} = \lambda^{(2)}\mathbf{x}^{(2)}$$

이므로

$$\lambda^{(2)} = -5\ , \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -\frac{2}{5} \end{bmatrix}$$

이다. 이러한 과정을 반복하면 그 결과는 다음과 같다.

반복횟수	λ	x_1	x_2
0	·	1.000000	1.000000
1	−3.000000	1.000000	0.000000
2	−5.000000	1.000000	−0.400000
3	−5.800000	1.000000	−0.482759
4	−5.965517	1.000000	−0.497110
5	−5.994220	1.000000	−0.499518
6	−5.999036	1.000000	−0.499920
7	−5.999839	1.000000	−0.499987
8	**−5.999973**	**1.000000**	**−0.499998**

따라서 수렴한 후 가장 큰 고유값과 고유벡터는 다음과 같다.

$$\lambda_1 = -6\ , \qquad \mathbf{x}_1 = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$$

Numerical Methods (II)

- **Inverse power method**

  - Iteration formula

$$A^{-1}x^{(k)} = y^{(k+1)} \Leftrightarrow Ay^{(k+1)} = x^{(k)}$$
$$\Leftrightarrow Lc = x^{(k)}, Uy^{(k+1)} = c$$

  - for obtaining small λ

Exploiting shifting property

Let the eigenvalues of $A = \lambda_1, \lambda_2, \cdots \lambda_n$

then, the eigenvalues of $(A - aI)$

$$= \lambda_1 - a, \lambda_2 - a, \cdots, \lambda_n - a,$$

- Finding the maximum eigenvalue with opposite sign after obtaining λ
- Accelerating the convergence when an approximate eigenvalue is available

Deflated matrices

Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A with associated eigenvectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(n)}$ and that λ_1 has multiplicity 1. If $\mathbf{x}$ is a vector with $\mathbf{x}^t\mathbf{v}^{(1)} = 1$, then

$$B = A - \lambda_1 \mathbf{v}^{(1)}\mathbf{x}^t$$

has eigenvalues $0, \lambda_2, \lambda_3, \ldots, \lambda_n$ with associated eigenvectors $\mathbf{v}^{(1)}, \mathbf{w}^{(2)}, \mathbf{w}^{(3)}, \ldots, \mathbf{w}^{(n)}$, where $\mathbf{v}^{(i)}$ and $\mathbf{w}^{(i)}$ are related by the equation

$$\mathbf{v}^{(i)} = (\lambda_i - \lambda_1)\mathbf{w}^{(i)} + \lambda_1(\mathbf{x}^t\mathbf{w}^{(i)})\mathbf{v}^{(1)},$$

for each $i = 2, 3, \ldots, n$.

- It is possible to obtain eigenvectors one after another
- Properly assigning the vector ***x*** is important
- Eg. Wielandt's deflation

$$\mathbf{x} = \frac{1}{\lambda_1 v_i^{(1)}}(a_{i1}, a_{i2}, \ldots, a_{in})^t,$$

Eg. Using Deflation(I)

Example 4 The symmetric matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

has eigenvalues $\lambda_1 = 6$, $\lambda_2 = 3$, and $\lambda_3 = 1$. Assuming that the dominant eigenvalue $\lambda_1 = 6$ and associated unit eigenvector $\mathbf{v}^{(1)} = (1, -1, 1)^t$ have been calculated, the procedure just outlined for obtaining λ_2 proceeds as follows:

$$\mathbf{x} = \frac{1}{6}\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} = \left(\frac{2}{3}, -\frac{1}{6}, \frac{1}{6}\right)^t,$$

$$\mathbf{v}^{(1)}\mathbf{x}^t = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\left[\tfrac{2}{3}, \quad -\tfrac{1}{6}, \quad \tfrac{1}{6}\right] = \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & \frac{1}{6} \\ -\frac{2}{3} & \frac{1}{6} & -\frac{1}{6} \\ \frac{2}{3} & -\frac{1}{6} & \frac{1}{6} \end{bmatrix},$$

and

$$B = A - \lambda_1\mathbf{v}^{(1)}\mathbf{x}^t = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix} - 6\begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & \frac{1}{6} \\ -\frac{2}{3} & \frac{1}{6} & -\frac{1}{6} \\ \frac{2}{3} & -\frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 2 & -1 \\ -3 & -1 & 2 \end{bmatrix}.$$

Eg. Using Deflation(II)

Deleting the first row and column gives

$$B' = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

which has eigenvalues $\lambda_2 = 3$ and $\lambda_3 = 1$. For $\lambda_2 = 3$, the eigenvector $\mathbf{w}^{(2)'}$ can be obtained by solving the second-order linear system

$$(B' - 3I)\mathbf{w}^{(2)'} = \mathbf{0}, \quad \text{resulting in} \quad \mathbf{w}^{(2)'} = (1, -1)^t.$$

Adding a zero for the first component gives $\mathbf{w}^{(2)} = (0, 1, -1)^t$ and

$$\mathbf{v}^{(2)} = (3-6)(0, 1, -1)^t + 6\left[\left(\frac{2}{3}, -\frac{1}{6}, \frac{1}{6}\right)(0, 1, -1)^t\right](1, -1, 1)^t$$
$$= (-2, -1, 1)^t.$$

Numerical Methods (III)

Hotelling's deflation method

- Iteration formula:

  $A_{i+1} = A_i - \lambda_i x_i x_i^T$ given λ_i, x_i
- for symmetric matrices
- deflation from large to small λ

Numerical Methods (IV)

Jacobi transformation

- Successive diagonalization without changing λ.
- for symmetric matrices

- Atomic transformation:

$$\mathbf{P}_{pq} = \begin{bmatrix} 1 & & & & & & \\ & \cdots & & & & & \\ & & c & \cdots & s & & \\ & & \vdots & 1 & \vdots & & \\ & & -s & \cdots & c & & \\ & & & & & \cdots & \\ & & & & & & 1 \end{bmatrix}$$

Homework

- Generate a 11x11 ***symmetric*** matrix ***A*** by using random number generator(Gaussian distribution with mean=0 and standard deviation=1.0]). Then, compute all eigenvalues and eigenvectors of ***A*** using the routines in the book, NR in C. Print the eigenvalues and their corresponding eigenvectors in the descending order.
  - You may use
    - jacobi(): Obtaining eigenvalues using the Jacobi transformation
    - eigsrt(): Sorting the results of jacobi()