Numerical Analysis – Digital Signal Processing

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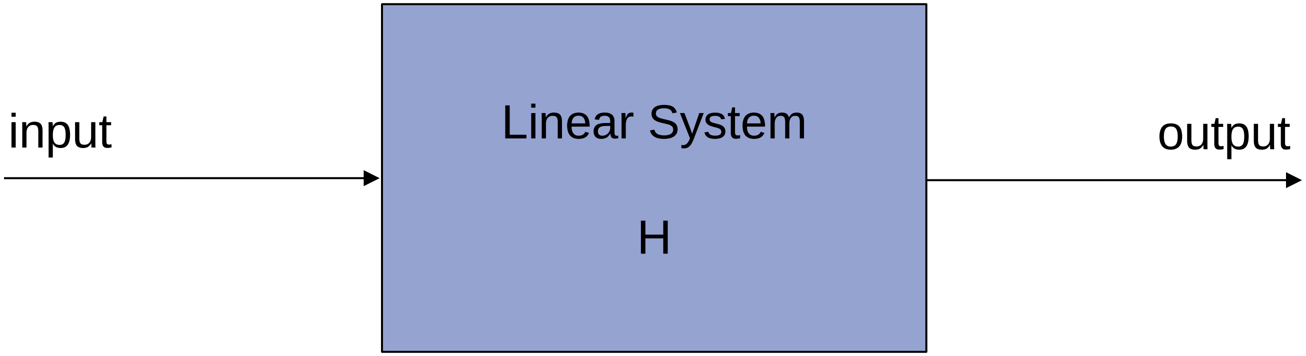
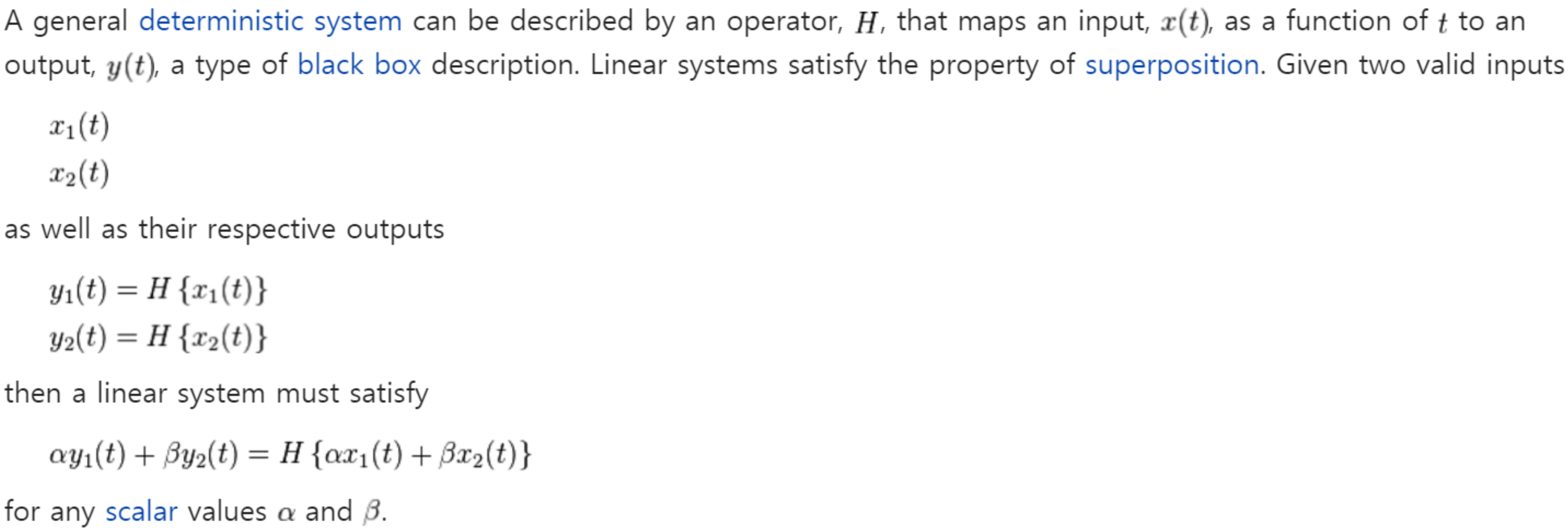
Digital Signal Processing

- Linear Systems
- Sampling and Reconstruction
- Convolution
- Discrete Fourier Transform
- Fast Fourier Transform(FFT)
- Multi-dimensional FFT

Linear Systems





A general deterministic system can be described by an operator, H, that maps an input, $x(t)$, as a function of t to an output, $y(t)$, a type of black box description. Linear systems satisfy the property of superposition. Given two valid inputs

$$x_1(t)$$
$$x_2(t)$$

as well as their respective outputs

$$y_1(t) = H\left\{x_1(t)\right\}$$
$$y_2(t) = H\left\{x_2(t)\right\}$$

then a linear system must satisfy

$$\alpha y_1(t) + \beta y_2(t) = H\left\{\alpha x_1(t) + \beta x_2(t)\right\}$$

for any scalar values α and β.

[from Wikipedia]

Notation and definitions

- One-dimensional signal
  - Continuous signal : $f(x), u(x), s(t), ...$
  - Sampled signal : $u_n, u(n), ...$
- Two-dimensional signal
  - Continuous signal : $u(x, y), v(x, y), f(x, y), ...$
  - Sampled signal : $u_{m,n}, v(m, n), u(i, j), ...$
    - $i, j, k, l, m, n,$ … are usually used to specify integer indices
  - Separable form : $f(x, y) = f(x) f(y)$

Delta function

- 2-D delta function
  - Dirac : $\delta(x,y)=\delta(x)\delta(y)$
    - Property

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')\delta(x-x',y-y')dx'dy'= f(x,y),$$

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\delta(x,y)dxdy=1$$

    - Scaling : $\delta(ax)=\delta(x)/|a|,$

$$\delta(ax,by)=\delta(x,y)/|ab|,$$

  - Kronecker delta : $\delta(m,n)=\delta(m)\delta(n)$
    - Property

$$x(m,n)=\sum_{m'=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}x(m',n')\delta(m-m',n-n'),\quad \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(m,n)=1$$

Special signals

- Some special signals(or functions)

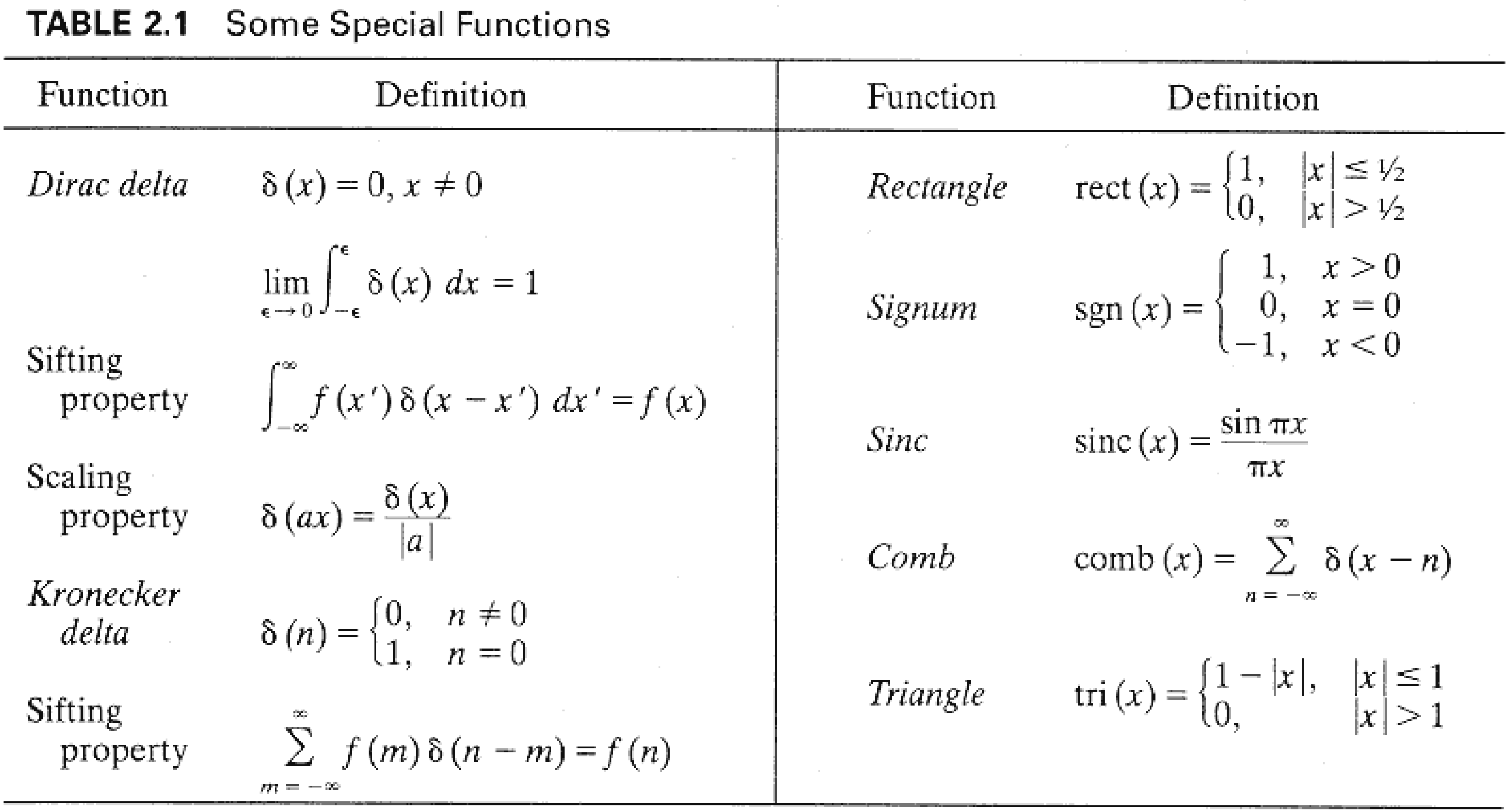


TABLE 2.1 Some Special Functions

Function	Definition	Function	Definition
Dirac delta	$\delta(x) = 0, x \neq 0$	*Rectangle*	$\mathrm{rect}(x) = \begin{cases} 1, & \lvert x \rvert \leq ½ \\ 0, & \lvert x \rvert > ½ \end{cases}$
	$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \delta(x)\, dx = 1$	*Signum*	$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$
Sifting property	$\int_{-\infty}^{\infty} f(x')\delta(x - x')\, dx' = f(x)$	*Sinc*	$\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$
Scaling property	$\delta(ax) = \frac{\delta(x)}{\lvert a \rvert}$	*Comb*	$\mathrm{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n)$
Kronecker delta	$\delta(n) = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$	*Triangle*	$\mathrm{tri}(x) = \begin{cases} 1 - \lvert x \rvert, & \lvert x \rvert \leq 1 \\ 0, & \lvert x \rvert > 1 \end{cases}$
Sifting property	$\sum_{m=-\infty}^{\infty} f(m)\delta(n - m) = f(n)$		

Linear and shift invariant systems

$$x(m,n) \longrightarrow \boxed{H[\bullet]} \longrightarrow y(m,n)=H[x(m,n)]$$

- Linearity

$$H[a_1x_1(m,n)+a_2x_2(m,n)]=a_1H[x_1(m,n)]+a_2H[x_2(m,n)]$$

$$=a_1y_1(m,n)+a_2y_2(m,n),\ for\ \forall a_1,a_2,x_1(\cdot),x_2(\cdot)$$

- Output of linear systems

$$y(m,n)=H[x(m,n)]=H[\sum_{m'}\sum_{n'}x(m',n')\delta(m-m',n-n')]$$

$$=\sum_{m'}\sum_{n'}x(m',n')\underline{H[\delta(m-m',n-n')]}$$

by superposition

impulse response, unit sample response,
point spread function(PSF)

- Definition of impulse response

$$h(m,n;m',n')\equiv H[\delta(m-m',n-n')]$$

Shift invariance

- Shift invariance

$$\left.\begin{aligned} &If \quad y(m,n) = H[x(m,n)] \quad and \\ &\quad y(m-m_0, n-n_0) = H[x(m-m_0, n-n_0)],\ for\ \forall m_0, n_0 \end{aligned}\right\}$$

definition of
shift invariance

$$then, \quad h(m,n;m_0,n_0) = h(m-m_0, n-n_0)$$

Output of LSI(linear shift invariant) systems

$$y(m,n) = \sum_{m'}\sum_{n'} x(m',n')h(m-m',n-n')$$ (2-D convolution)

$$\because \quad y(m,n) = H[x(m,n)] = H[\sum_{m'}\sum_{n'} x(m',n')\delta(m-m',n-n')]$$

by superposition
of linearity

$$= \sum_{m'}\sum_{n'} x(m',n')H[\delta(m-m',n-n')]$$

by definition of
impulse response

$$= \sum_{m'}\sum_{n'} x(m',n')h(m,n;m',n')$$

by shift invariance

$$= \sum_{m'}\sum_{n'} x(m',n')h(m-m',n-n')$$

Stability

- Stability
  - Definition : bounded input, bounded output

$$if\ |x(m,n)| < \infty, \quad then\ |H[x(m,n)]| < \infty$$

  - Stable LSI systems(necessary and sufficient condition)

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}|h(m,n)| < \infty$$

The Fourier transform

- Definition
  - 1-D Fourier transform

$$f(x)=\int_{-\infty}^{\infty}F(u)\exp(j2\pi ux)du$$

$$\Leftrightarrow \quad F(u)=\int_{-\infty}^{\infty}f(x)\exp(-j2\pi ux)dx$$

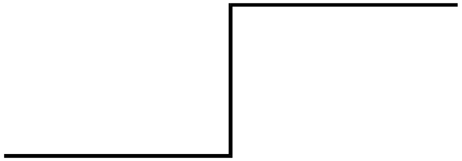
  - 2-D Fourier transform

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)\exp(j2\pi(xu+yv))dudv$$

$$\Leftrightarrow \quad F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\exp(-j2\pi(ux+vy))dxdy$$

Frequency domain

- Properties
  - $f(t) \rightarrow F(\omega)$; ω = angular frequency
  - $f(x,y) \rightarrow F(u,v)$; u,v = spatial frequencies that represent the luminance change with respect to spatial distance



edge

→ high spatial frequencies

Properties of Fourier transform

- Uniqueness
  - $f(x,y)$ and $F(u,v)$ are unique with respect to one another
- Separability

$$F(u,v)=\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} f(x,y)\exp(-j2\pi ux)dx]\exp(-j2\pi vy)dy$$

- Convolution theorem

$$g(x,y)=h(x,y)*f(x,y) \quad \Longleftrightarrow \quad G(u,v)=H(u,v)F(u,v)$$

- Inner product preservation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)h^*(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)H^*(u,v)dudv$$

Setting *h=f, Parseval energy conservation formula*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} | f(x,y)|^2 \, dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} | F(u,v)|^2 \, dudv$$

- Hankel transform : polar coordinate form of FT

$$F_p(\xi,\phi) \equiv F(\xi\cos\phi, \xi\sin\phi)$$

$$= \int_0^{2\pi}\int_0^{\infty} f_p(r,\theta)\exp[-j2\pi r\xi\cos(\theta-\phi)]rdrd\theta$$

where $f_p(r,\theta) = f(r\cos\theta, r\sin\theta)$

Fourier series

- 1-D case

$$x(n) = \int_{-0.5}^{0.5} X(u)\exp(-j2\pi nu)du$$

$$\Leftrightarrow \quad X(u) = \sum_{n=-\infty}^{\infty} x(n)\exp(-j2\pi nu), \quad -0.5 \le u < 0.5$$

2D Fourier series

- 2-D case

$$x(m,n)=\int_{-0.5}^{0.5}\int_{-0.5}^{0.5}X(u,v)\exp(j2\pi(mu_1+nv))dudv$$

$$\Leftrightarrow\ X(u,v)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}x(m,n)\exp(-j2\pi(mu+nv)),\quad -0.5\le u,v<0.5$$

  - $X(u,v)$ is periodic : period = 1

$$X(u,v)=X(u\pm k,v\pm l),\ \ k,l=0,1,2,\cdots$$

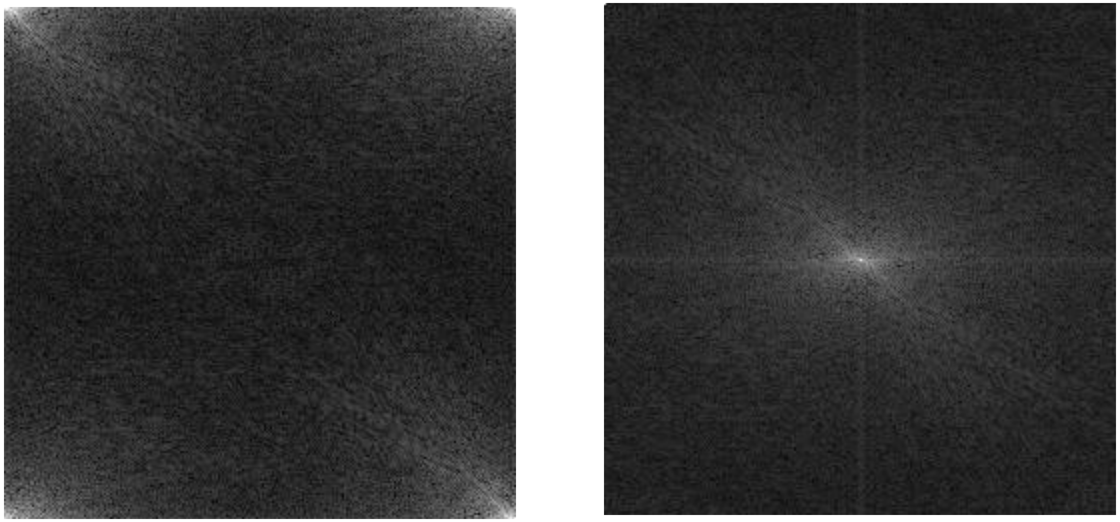
  - Sufficient condition for existence of $X(u,v)$

$$|X(u,v)|=|\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}x(m,n)\exp(-j2\pi(mu+nv))|$$

$$\le\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}|x(m,n)||\exp(-j2\pi(mu+nv))|\le\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}|x(m,n)|<\infty$$

Eg. 2D Fourier transform

original 256x256 lena



normalized spectrum
(log-scale)

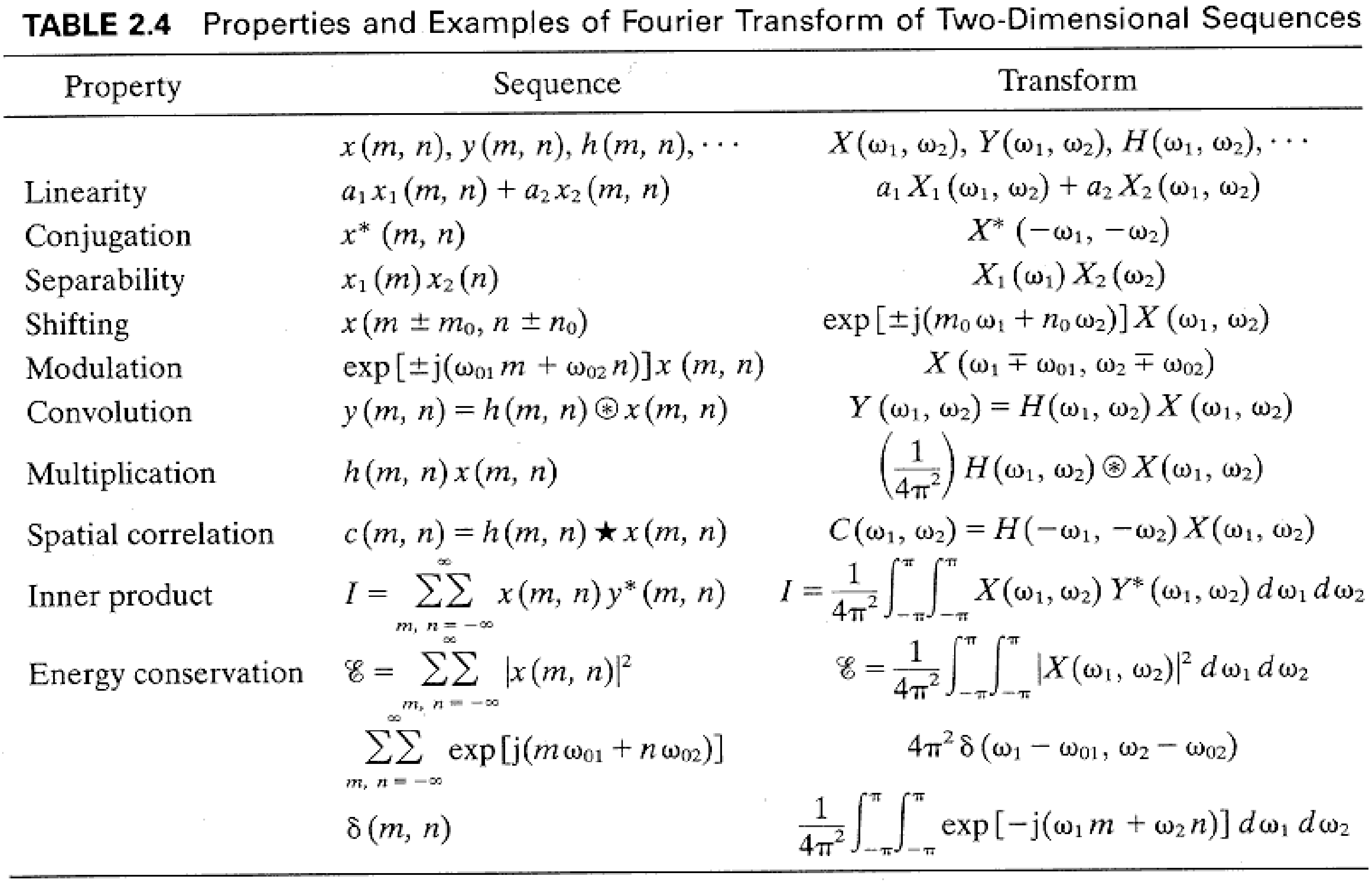
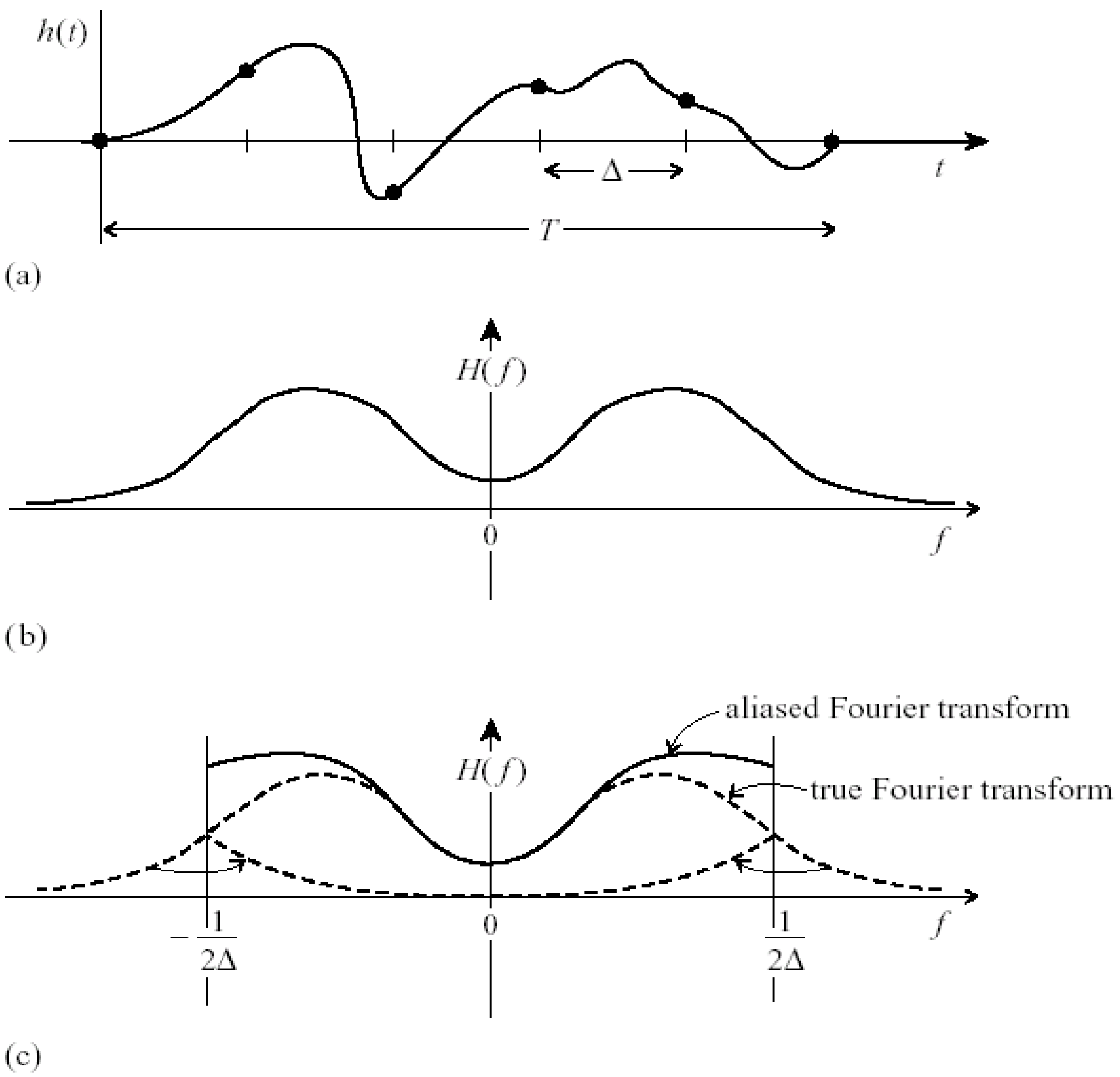


TABLE 2.4 Properties and Examples of Fourier Transform of Two-Dimensional Sequences

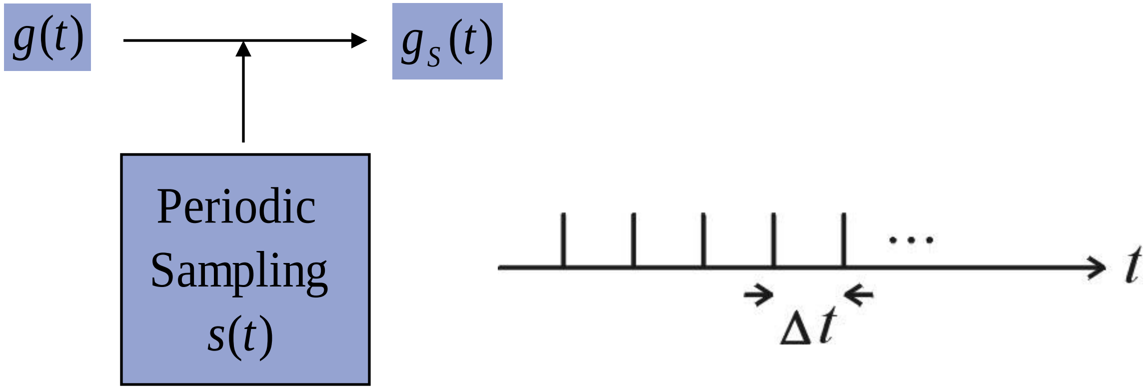
Property	Sequence	Transform
	$x(m, n), y(m, n), h(m, n), \cdots$	$X(\omega_1, \omega_2), Y(\omega_1, \omega_2), H(\omega_1, \omega_2), \cdots$
Linearity	$a_1 x_1(m, n) + a_2 x_2(m, n)$	$a_1 X_1(\omega_1, \omega_2) + a_2 X_2(\omega_1, \omega_2)$
Conjugation	$x^*(m, n)$	$X^*(-\omega_1, -\omega_2)$
Separability	$x_1(m) x_2(n)$	$X_1(\omega_1) X_2(\omega_2)$
Shifting	$x(m \pm m_0, n \pm n_0)$	$\exp[\pm j(m_0 \omega_1 + n_0 \omega_2)] X(\omega_1, \omega_2)$
Modulation	$\exp[\pm j(\omega_{01} m + \omega_{02} n)] x(m, n)$	$X(\omega_1 \mp \omega_{01}, \omega_2 \mp \omega_{02})$
Convolution	$y(m, n) = h(m, n) \circledast x(m, n)$	$Y(\omega_1, \omega_2) = H(\omega_1, \omega_2) X(\omega_1, \omega_2)$
Multiplication	$h(m, n) x(m, n)$	$\left(\frac{1}{4\pi^2}\right) H(\omega_1, \omega_2) \circledast X(\omega_1, \omega_2)$
Spatial correlation	$c(m, n) = h(m, n) \star x(m, n)$	$C(\omega_1, \omega_2) = H(-\omega_1, -\omega_2) X(\omega_1, \omega_2)$
Inner product	$I = \sum\sum_{m, n = -\infty}^{\infty} x(m, n) y^*(m, n)$	$I = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_1, \omega_2) Y^*(\omega_1, \omega_2)\, d\omega_1\, d\omega_2$
Energy conservation	$\mathscr{E} = \sum\sum_{m, n = -\infty}^{\infty} \lvert x(m, n) \rvert^2$	$\mathscr{E} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \lvert X(\omega_1, \omega_2) \rvert^2\, d\omega_1\, d\omega_2$
	$\sum\sum_{m, n = -\infty}^{\infty} \exp[j(m\omega_{01} + n\omega_{02})]$	$4\pi^2 \delta(\omega_1 - \omega_{01}, \omega_2 - \omega_{02})$
	$\delta(m, n)$	$\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \exp[-j(\omega_1 m + \omega_2 n)]\, d\omega_1\, d\omega_2$

Sampling and aliasing



Sampling Theory

- For One-Dimensional Signal



$g(t)$ → $g_S(t)$

Periodic Sampling $s(t)$

$$g_S(t) = g(t) \bullet s(t)$$

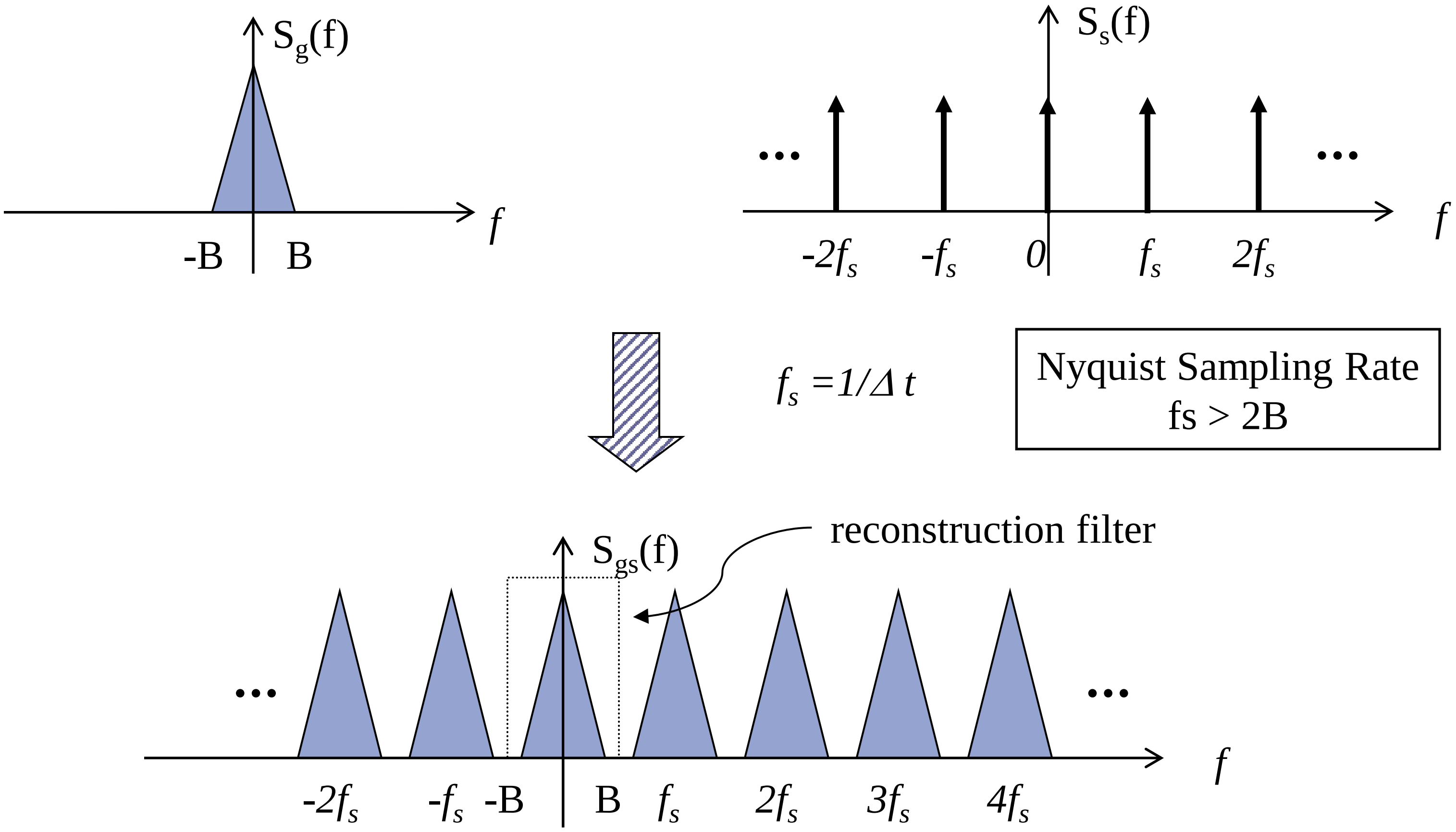
$$where \;\; s(t) = \sum_m \delta(t - m\Delta t)$$

$$\Rightarrow \quad S_{g_S}(f) = S_g(f) \otimes S_S(f)$$

(Fourier Transform)

Sampling Theory (cont.)

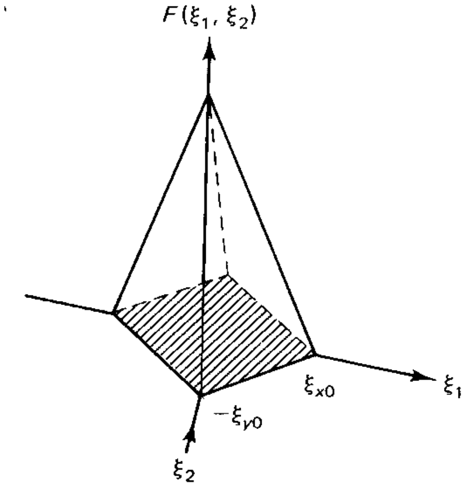
- For One-Dimensional Signal (cont.)



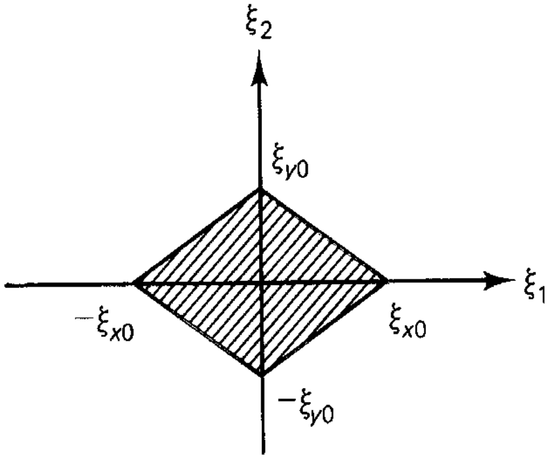
Nyquist Sampling Rate
fs > 2B

Sampling Theory(cont.)

- For Two-Dimensional Signal
  - Band-limited Image



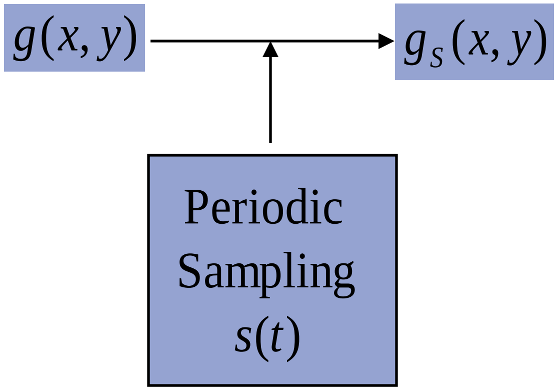
Fourier Transform of
a bandlimited function



Its region of support

Sampling Theory (cont.)

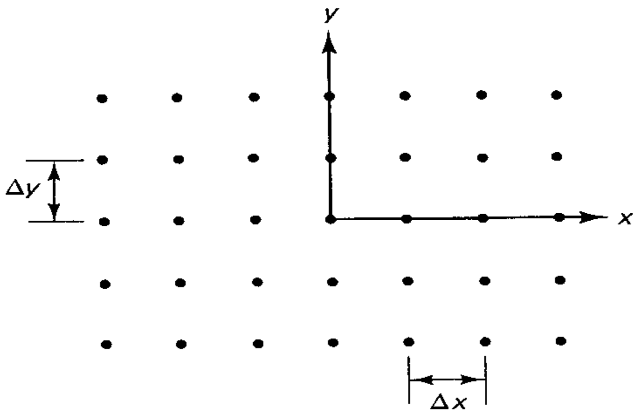
- For Two-Dimensional Signal (cont.)
  - Structure
    - Orthogonal Structure (Rectangular Tesselation)
    - Field Quincunx Structure (Triangular Tesselation)



$g(x,y)$

$g_S(x,y)$

Periodic Sampling $s(t)$

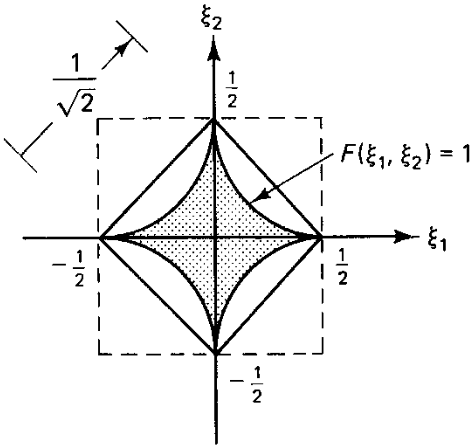


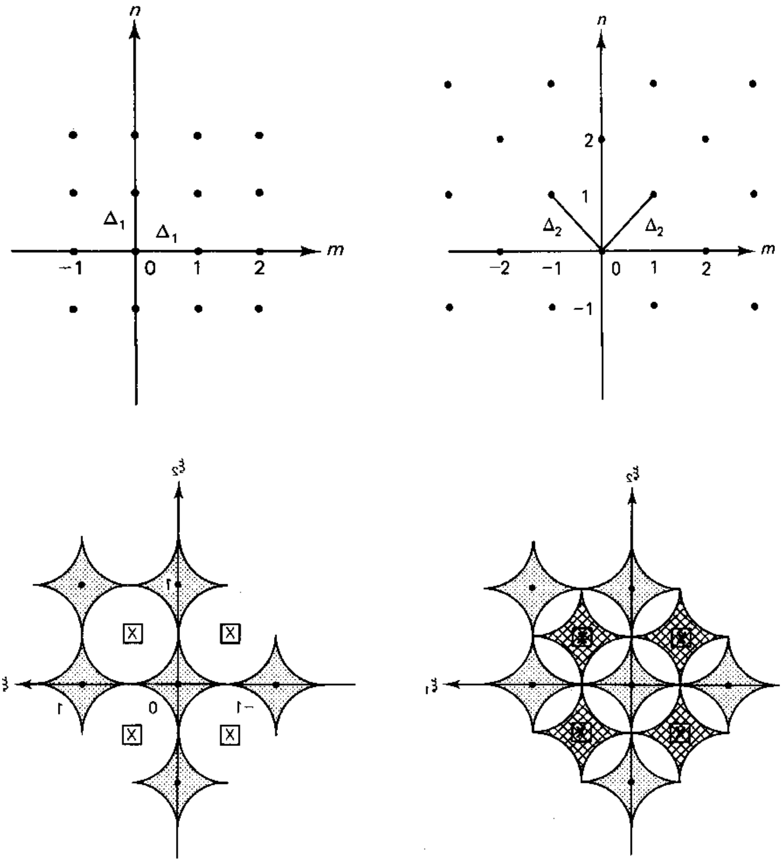
$$g_S(x,y) = g(x,y) \bullet s(x,y) \quad \Rightarrow \quad S_{g_S}(u,v) = S_g(u,v) \otimes S_S(u,v)$$

(Fourier Transform)

Sampling Theory(cont.)

- For Two-Dimensional Signal(cont.)
  - Structure(cont.)





2D sampling

- For Two-Dimensional Signal (cont.)
  - Orthogonal Structure (Rectangular Tesselation)

$$g_S(x,y) = g(x,y) \bullet s(x,y) \Longrightarrow S_{g_S}(u,v) = S_g(u,v) \otimes S_S(u,v)$$

(Fourier Transform)

- Sampling Function

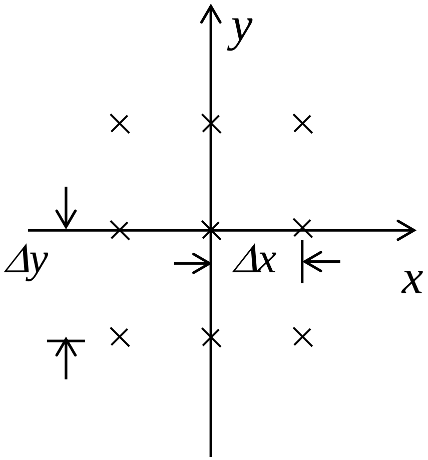
$$s(x,y) = \sum_n \sum_m \delta(x - m\Delta x, y - n\Delta y)$$

$$S_s(u,v) = \Delta u \Delta v \sum_k \sum_l \delta(u - k\Delta u, v - l\Delta v)$$

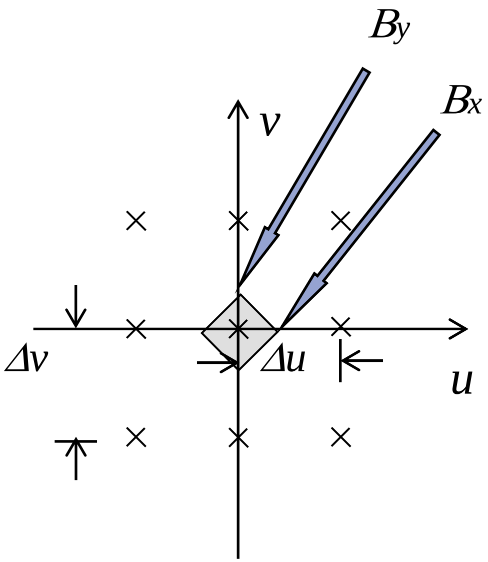
$$where \quad \Delta u = \frac{1}{\Delta x}, \quad \Delta v = \frac{1}{\Delta y}$$

2D sampling - Spectrum

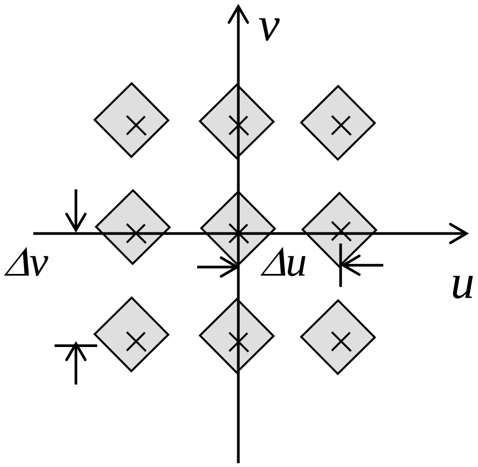
- For Two-Dimensional Signal (cont.)
  - Orthogonal Structure (cont.)
    - Spectrum of sampled signals



(a) sampling function



(b) spectrum of sampling function and signal



(c) spectrum of sampled signal

2D sampling - Reconstruction

- For Two-Dimensional Signal(cont.)
  - Orthogonal Structure(cont.)
    - Reconstruction of the original image from its samples

$$If\ \ 2B_x < \Delta u \quad and \quad 2B_y < \Delta v, \quad then$$

$$H(u,v) = \begin{cases} \Delta x \Delta y = \dfrac{1}{\Delta u \Delta v}, & (u,v) \in \Re \\ 0 \quad , & \text{otherwise} \end{cases}$$

$$g(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} g_S(m,n) \left(\frac{\sin(x\Delta u - m)\pi}{(x\Delta u - m)\pi} \right) \left(\frac{\sin(y\Delta v - n)\pi}{(y\Delta v - n)\pi} \right)$$

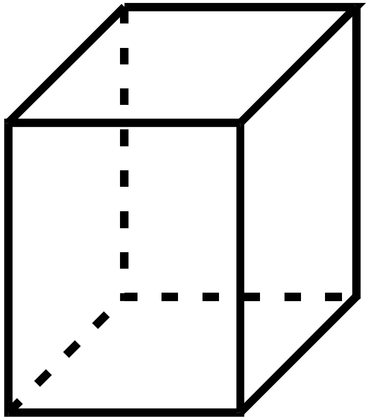
    - Nyquist Sampling Rate(or Frequency) and Nyquist Interval

$$\Delta u_{Nyquist} = 2B_x \,, \Delta v_{Nyquist} = 2B_y \,,$$

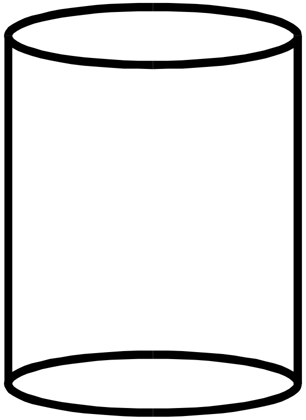
$$\Delta x_{Nyquist} = \frac{1}{\Delta u_{Nyquist}} = \frac{1}{2B_x} \,, \Delta y_{Nyquist} = \frac{1}{\Delta v_{Nyquist}} = \frac{1}{2B_y} \,,$$

Reconstruction Filter

rectangular filter



cylindrical filter



$$h_r(x,y) = K\left(\frac{\sin \omega_x x}{\omega_x x}\right)\left(\frac{\sin \omega_y y}{\omega_y y}\right)$$

$$h_c(x,y) = \frac{2\pi\omega_0 J_1\left(\omega_0\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}$$

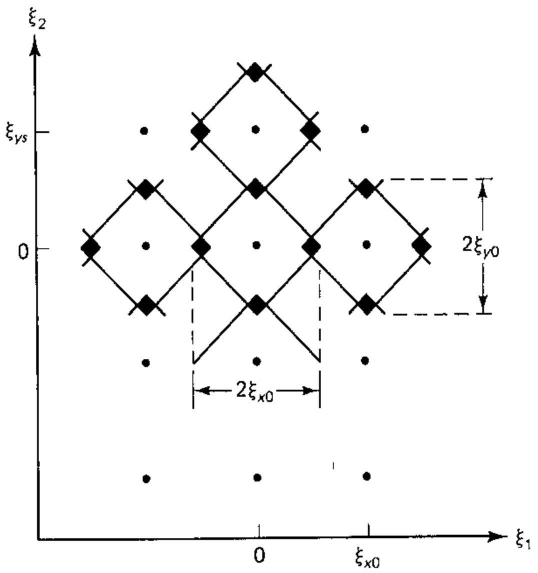
where $J_1(\cdot)$ is a first-order Bessel function

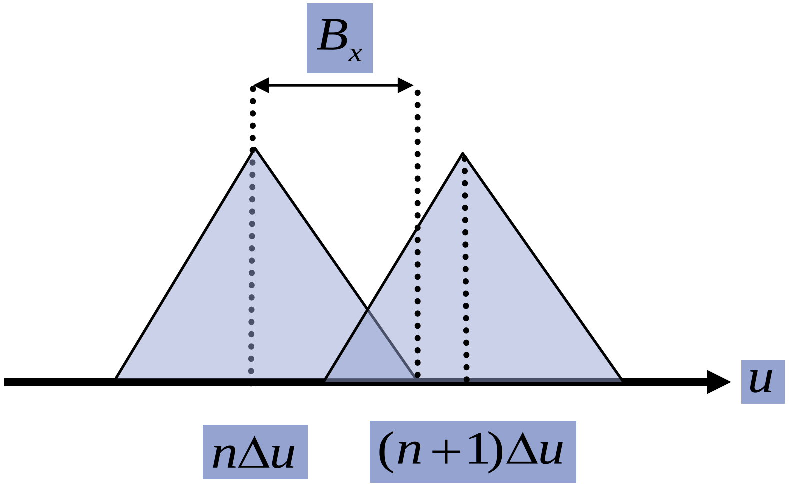
Aliasing effect

- For Two-Dimensional Signal(cont.)
  - Orthogonal Structure(cont.)
    - Aliasing Effect

$$\text{If} \quad \Delta u < \Delta u_{Nyquist} \quad \text{or} \quad \Delta v < \Delta v_{Nyquist}, \quad \text{then}$$

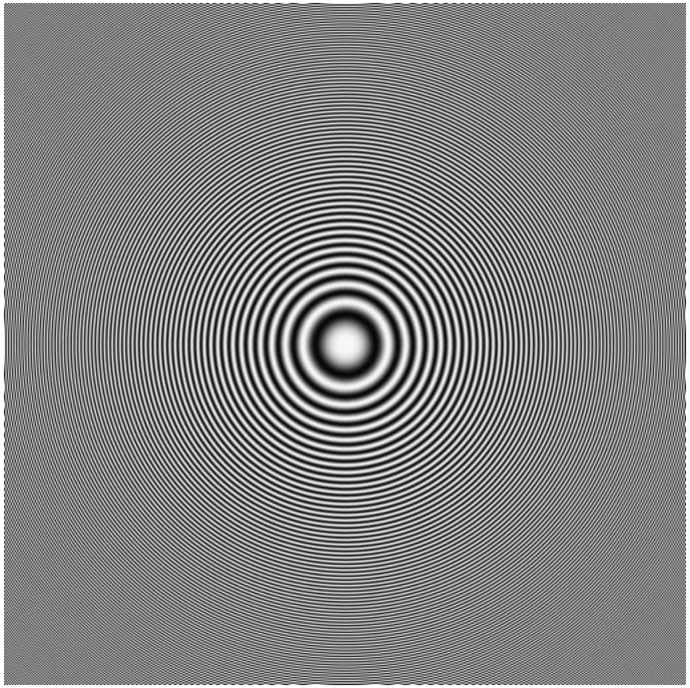
the original image cannot be reconstructed from its samples.



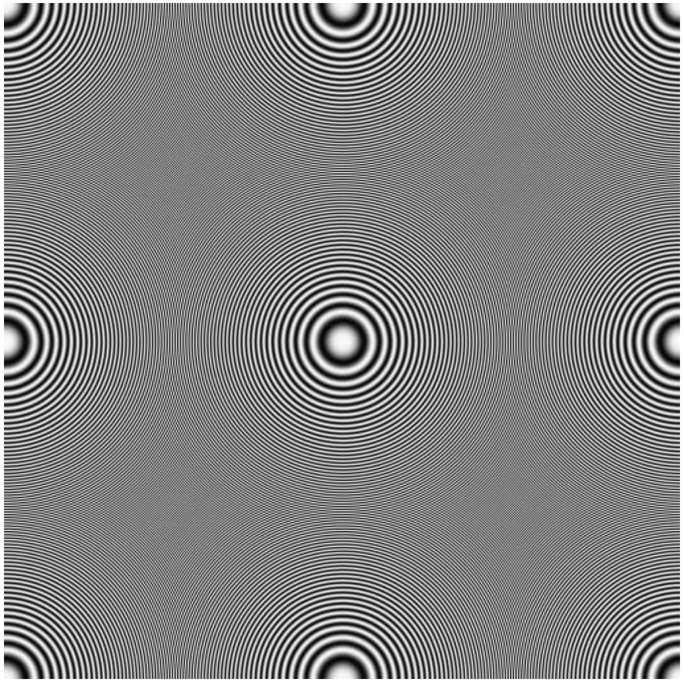


Eg. Aliasing

- For Two-Dimensional Signal (cont.)
  - Orthogonal Structure (cont.)
    - Aliasing Effect (cont.)



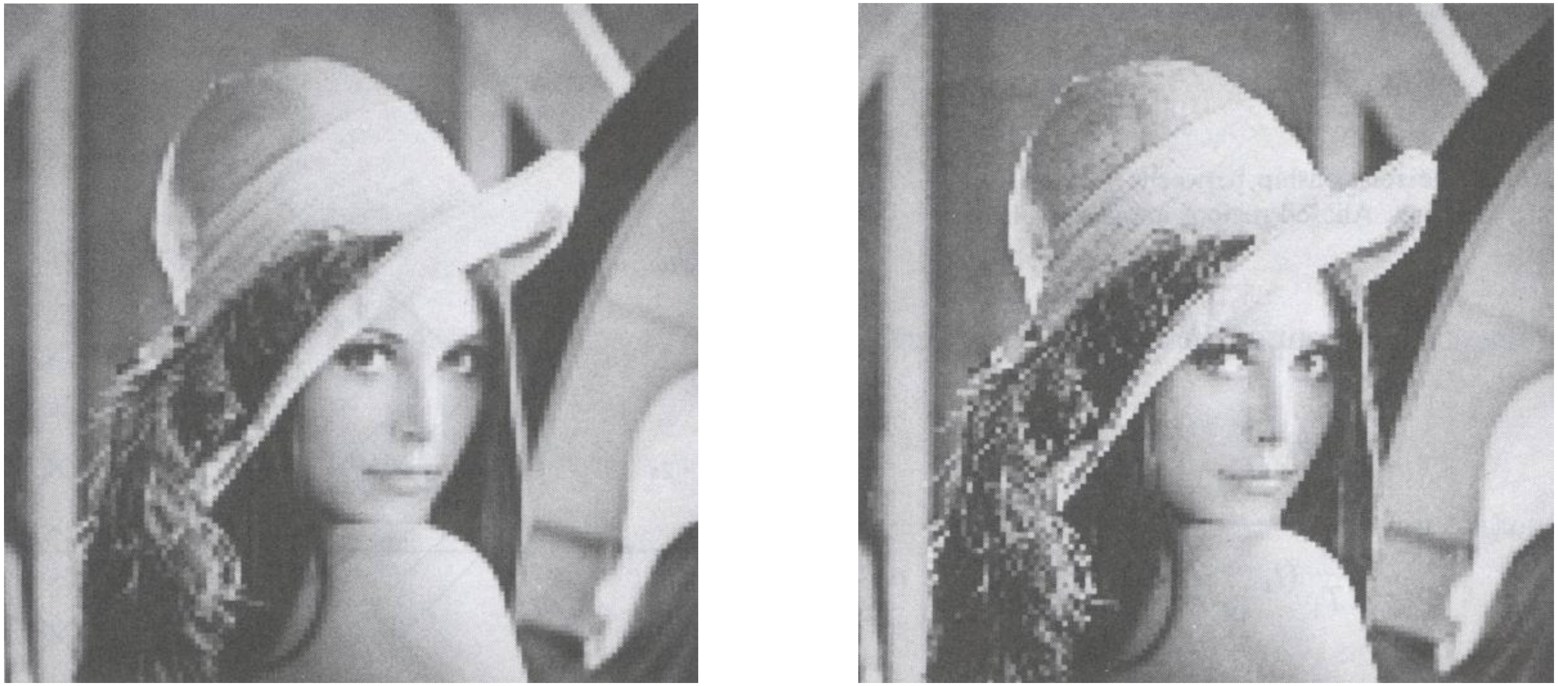
Zone Plate image ($\alpha = 1$)



Aliasing ($\alpha = 2$)

Eg. Aliasing

- Examples



Little aliasing due to an effective antialiasing filter

Noticeable aliasing

Practical limitations in sampling

- Practical Limitations
  - Real-world images are not band-limited.

    aliasing errors

    can be reduced by LPF before sampling

    LPF attenuate higher spatial frequencies

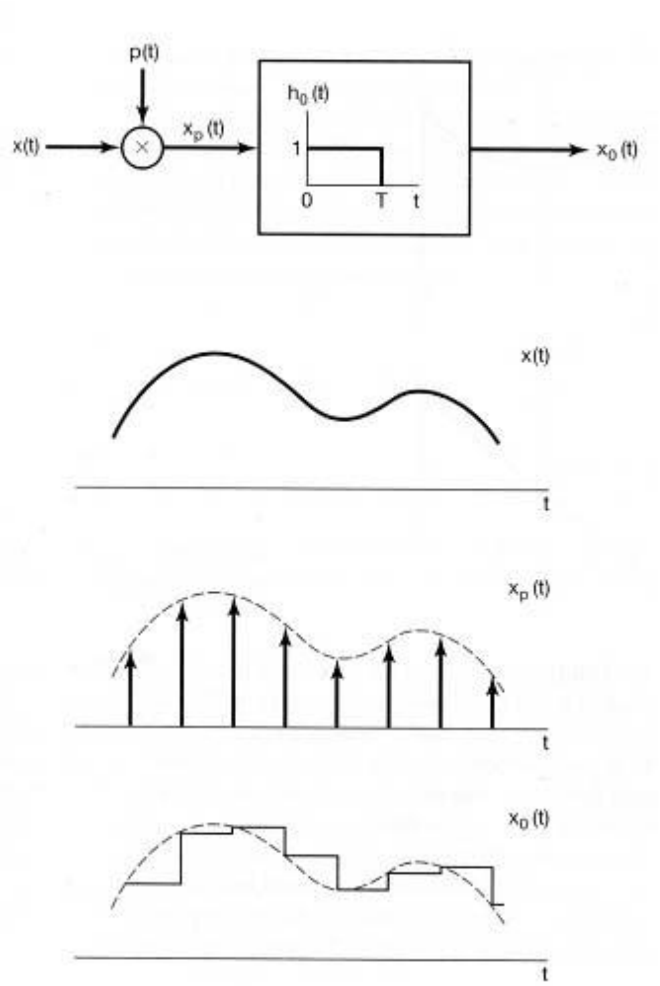
    Resolution loss

    blurring
  - No ideal LPF at reconstruction stage.

Sampling aperture

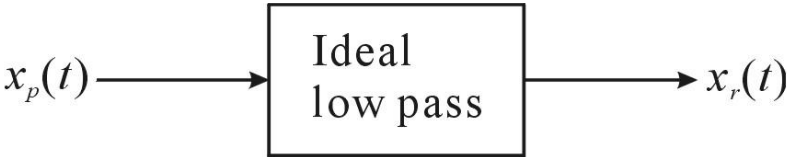
- Finite aperture (finite duration pulse)

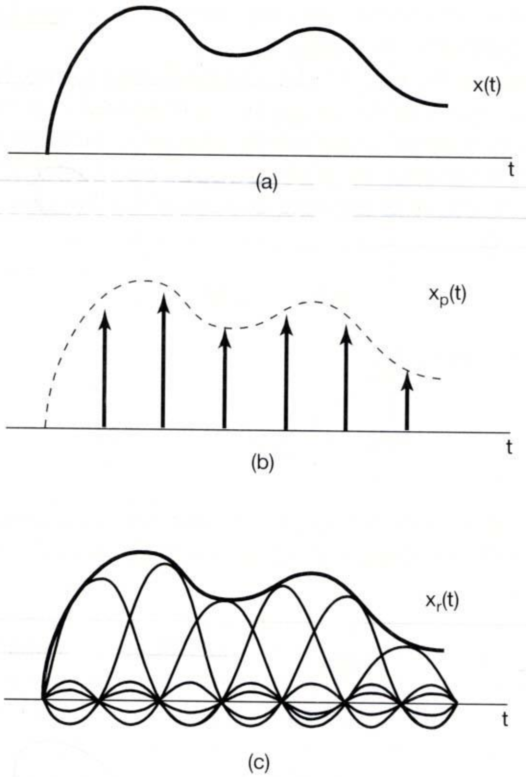
$$H_0(\omega) = e^{-j\omega T/2}\left[\frac{2\sin\left(\omega\, T/2\right)}{\omega}\right]$$



Reconstruction

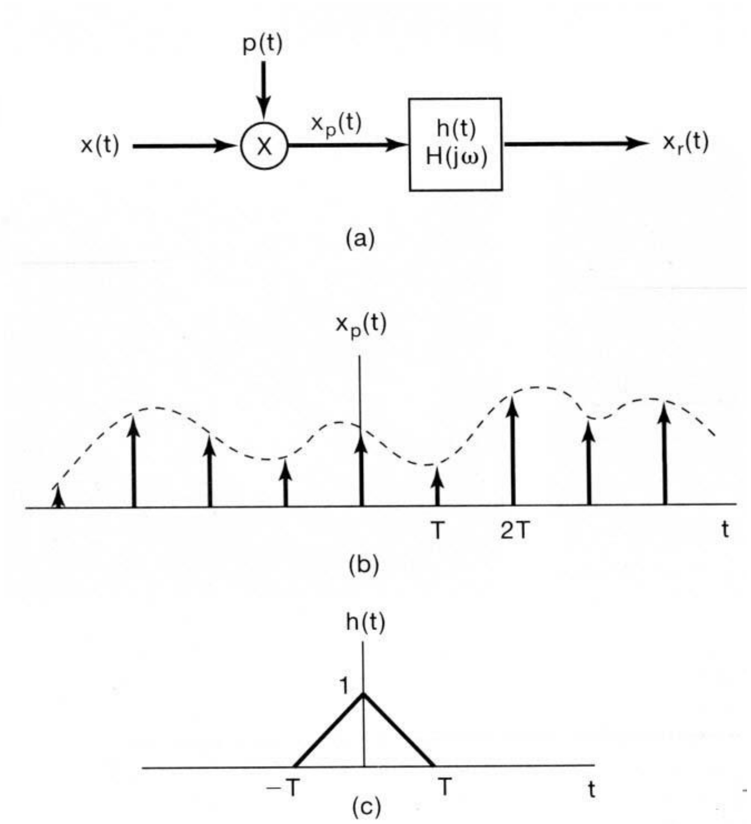
- Reconstruction of a signal from its sample using interpolation





Linear interpolation

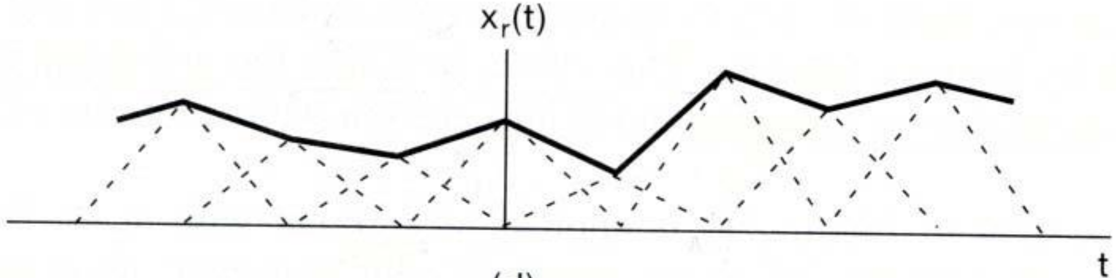
- linear interpolation



(a)

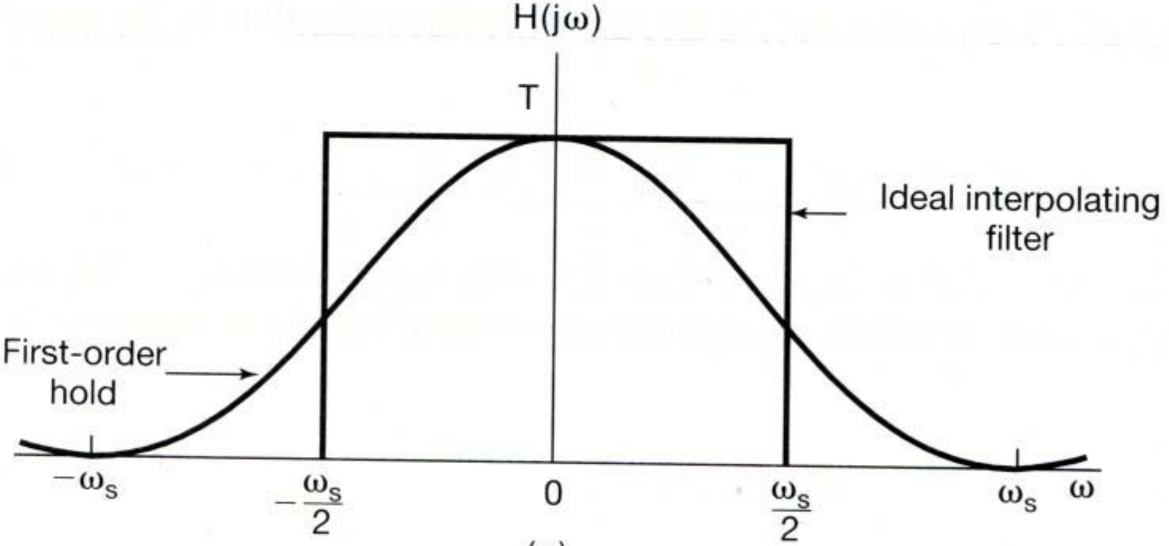
(b)

(c)



$x_r(t)$
t

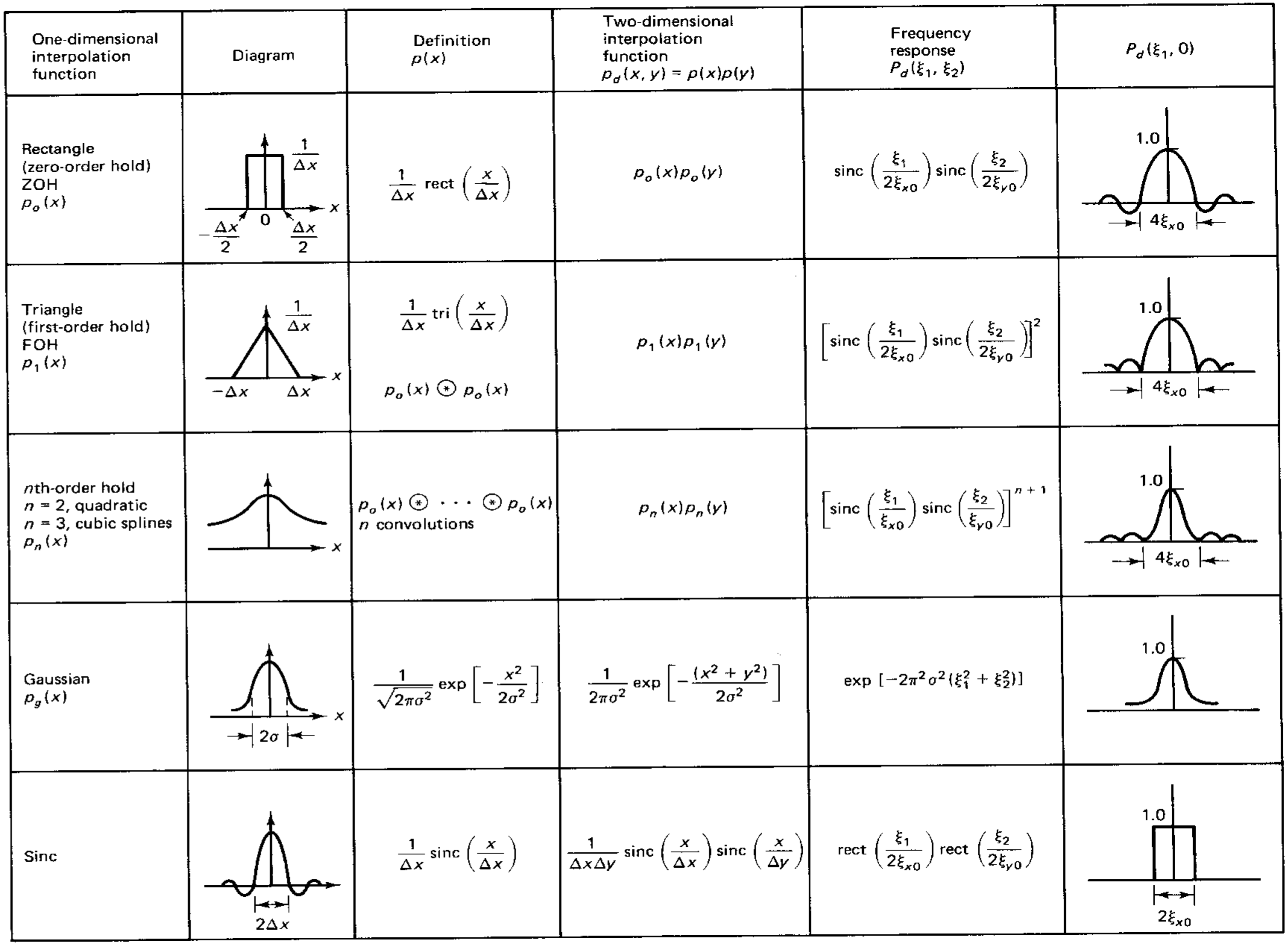
(d)



$H(j\omega)$
T
Ideal interpolating filter
First-order hold
$-\omega_s$
$-\frac{\omega_s}{2}$
0
$\frac{\omega_s}{2}$
ω_s
ω

(e)

Sampling Theory(cont.)

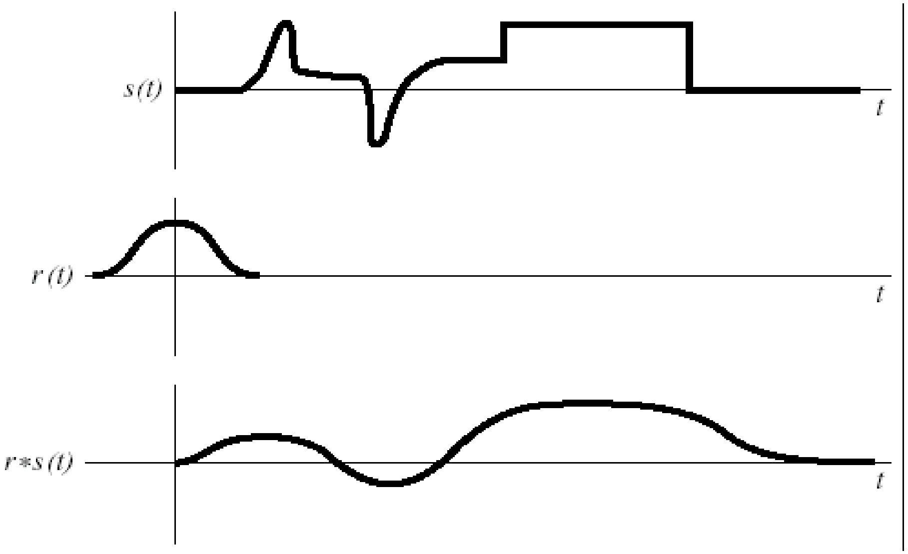


One-dimensional interpolation function	Diagram	Definition $p(x)$	Two-dimensional interpolation function $p_d(x, y) = p(x)p(y)$	Frequency response $P_d(\xi_1, \xi_2)$	$P_d(\xi_1, 0)$
Rectangle (zero-order hold) ZOH $p_o(x)$	$\frac{1}{\Delta x}$ x $-\frac{\Delta x}{2}$ 0 $\frac{\Delta x}{2}$	$\frac{1}{\Delta x}\operatorname{rect}\left(\frac{x}{\Delta x}\right)$	$p_o(x)p_o(y)$	$\operatorname{sinc}\left(\frac{\xi_1}{2\xi_{x0}}\right)\operatorname{sinc}\left(\frac{\xi_2}{2\xi_{y0}}\right)$	1.0 $4\xi_{x0}$
Triangle (first-order hold) FOH $p_1(x)$	$\frac{1}{\Delta x}$ x $-\Delta x$ Δx	$\frac{1}{\Delta x}\operatorname{tri}\left(\frac{x}{\Delta x}\right)$ $p_o(x)\circledast p_o(x)$	$p_1(x)p_1(y)$	$\left[\operatorname{sinc}\left(\frac{\xi_1}{2\xi_{x0}}\right)\operatorname{sinc}\left(\frac{\xi_2}{2\xi_{y0}}\right)\right]^2$	1.0 $4\xi_{x0}$
nth-order hold $n = 2$, quadratic $n = 3$, cubic splines $p_n(x)$	x	$p_o(x)\circledast\cdots\circledast p_o(x)$ n convolutions	$p_n(x)p_n(y)$	$\left[\operatorname{sinc}\left(\frac{\xi_1}{\xi_{x0}}\right)\operatorname{sinc}\left(\frac{\xi_2}{\xi_{y0}}\right)\right]^{n+1}$	1.0 $4\xi_{x0}$
Gaussian $p_g(x)$	x 2σ	$\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{x^2}{2\sigma^2}\right]$	$\frac{1}{2\pi\sigma^2}\exp\left[-\frac{(x^2+y^2)}{2\sigma^2}\right]$	$\exp\left[-2\pi^2\sigma^2(\xi_1^2+\xi_2^2)\right]$	1.0
Sinc	$2\Delta x$	$\frac{1}{\Delta x}\operatorname{sinc}\left(\frac{x}{\Delta x}\right)$	$\frac{1}{\Delta x\Delta y}\operatorname{sinc}\left(\frac{x}{\Delta x}\right)\operatorname{sinc}\left(\frac{x}{\Delta y}\right)$	$\operatorname{rect}\left(\frac{\xi_1}{2\xi_{x0}}\right)\operatorname{rect}\left(\frac{\xi_2}{2\xi_{y0}}\right)$	1.0 $2\xi_{x0}$

Convolution(I)

- Def.

$$g * h \equiv \int_{-\infty}^{\infty} g(\tau)h(t-\tau)\,d\tau$$



Convolution(II)

- Convolution theorem

$$\boxed{g * h \Longleftrightarrow G(f)H(f)}$$

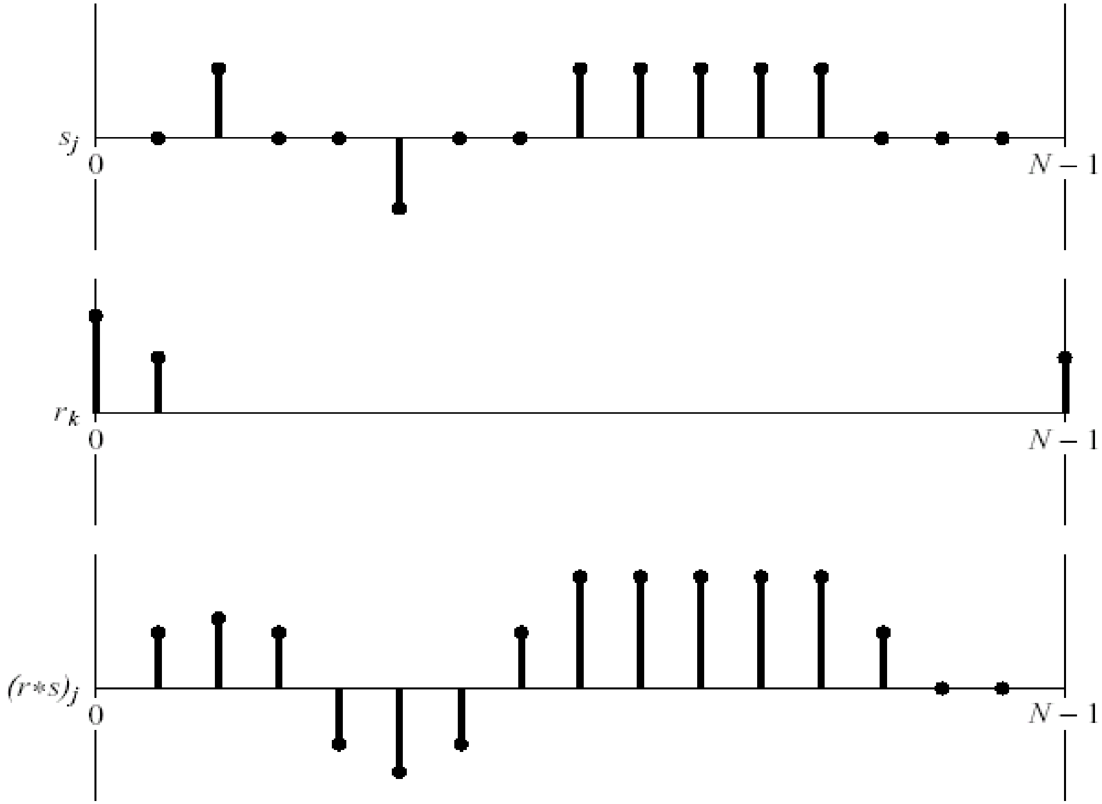
o direct convolution→ complex computation

o FFT and multiplication→ less computation

Convolution(III)

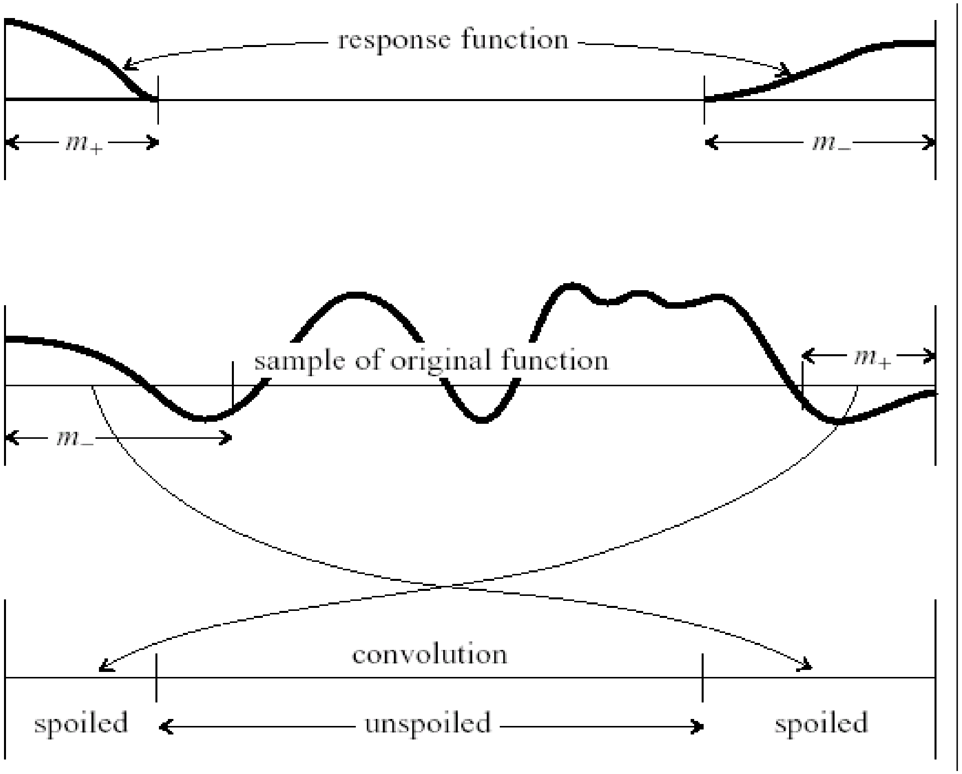
- Convolution of discrete sampled function

$$(r * s)_j \equiv \sum_{k=-M/2+1}^{M/2} s_{j-k}\, r_k$$



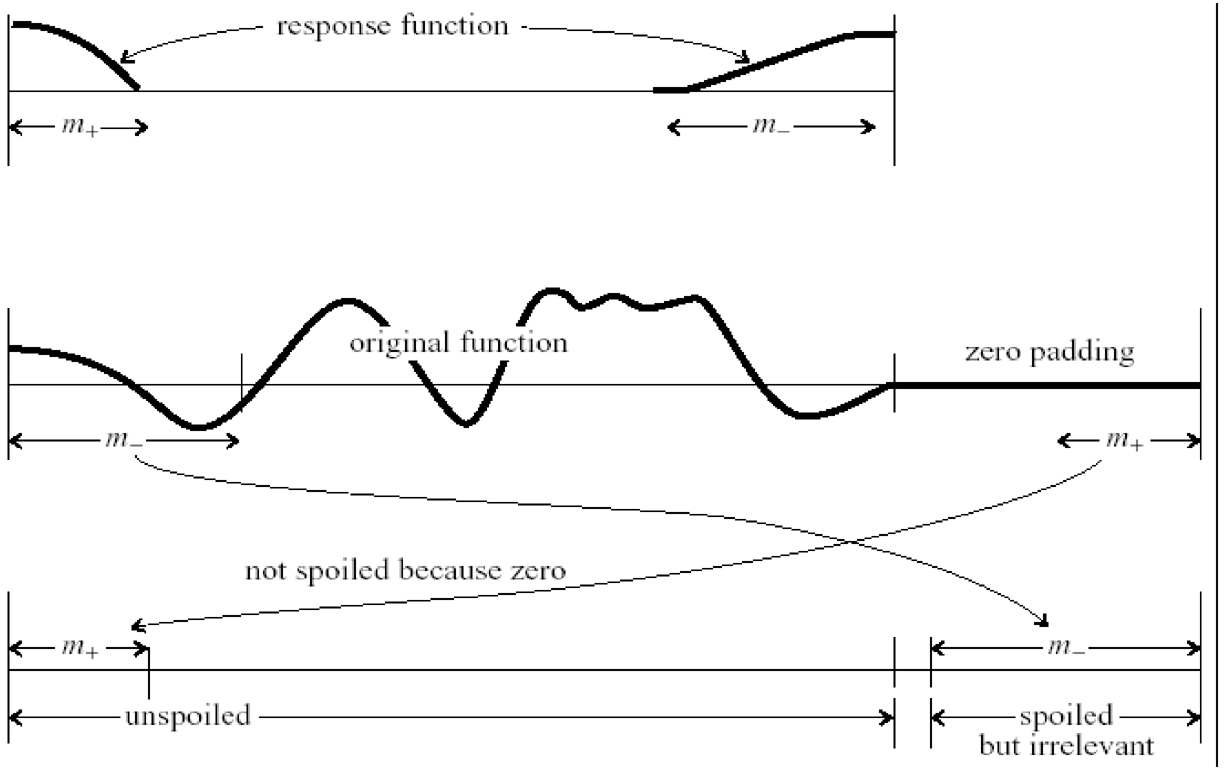
Convolution(IV)

- Trouble in using DFT of finite duration
  - ➔ End effects
  - ➔ Treated by zero padding
- End effect



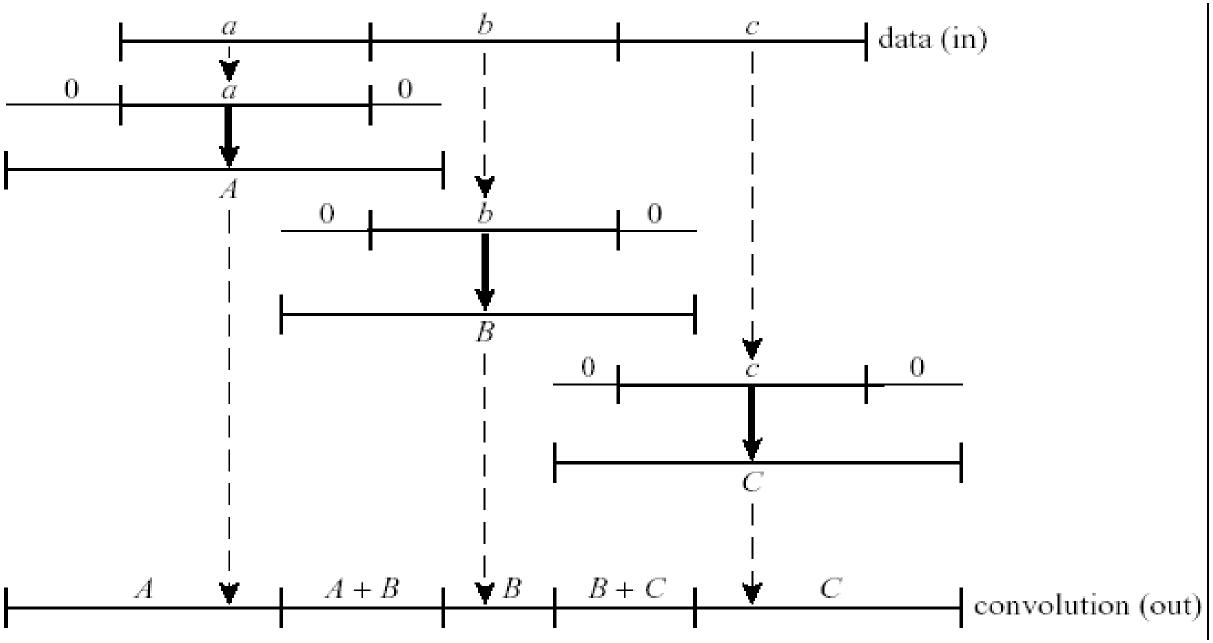
Convolution(V)

- Zero padding



Convolution(VI)

- Convolving very large data sets

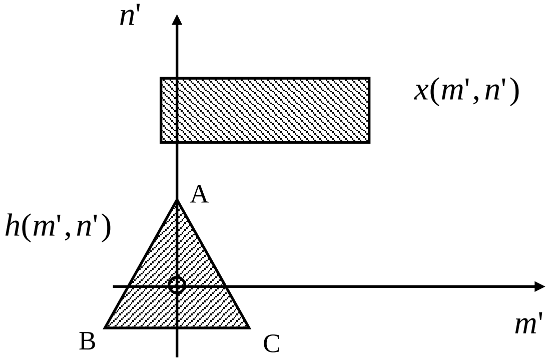


<Overlap-add method>

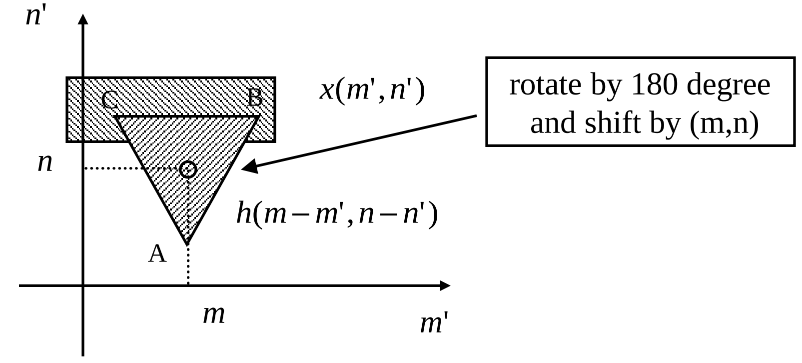
2D convolution

2-D convolution

$$y(m,n) = h(m,n) * x(m,n) = \sum_{m'} \sum_{n'} x(m',n') h(m-m', n-n')$$



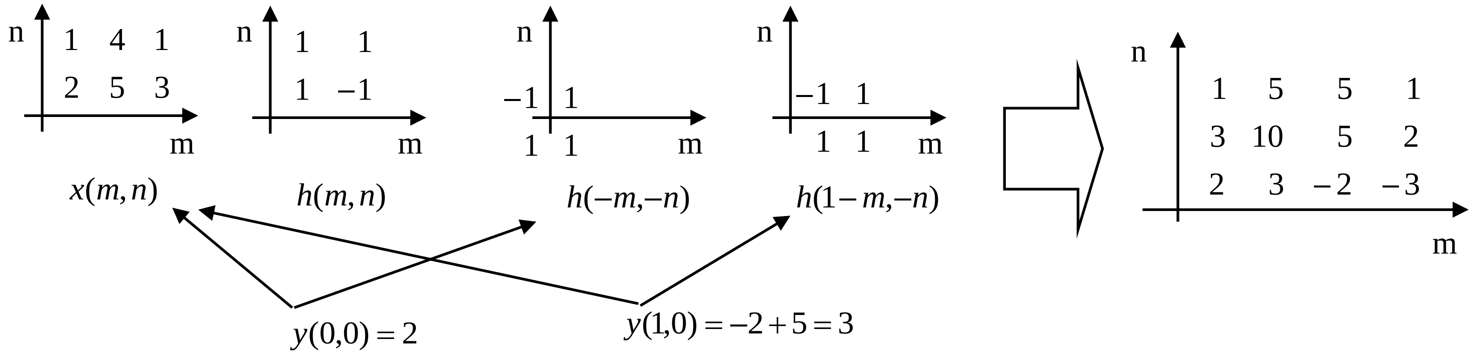
(a) impulse response



rotate by 180 degree and shift by (m,n)

(b) output at location (m,n) is the sum of product of quantities in the area of overlap

(ex)



$x(m,n)$

1	4	1
2	5	3

$h(m,n)$

1	1
1	−1

$h(-m,-n)$

−1	1
1	1

$h(1-m,-n)$

−1	1
1	1

1	5	5	1
3	10	5	2
2	3	−2	−3

$y(0,0) = 2$

$y(1,0) = -2 + 5 = 3$

Discrete Fourier Transform

- Fourier Transform

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f t} dt$$

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{-2\pi i f t} df$$

- Discrete Fourier Transform

$$H(f_n) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f_n t} dt \approx \sum_{k=0}^{N-1} h_k \, e^{2\pi i f_n t_k} \Delta = \Delta \sum_{k=0}^{N-1} h_k \, e^{2\pi i k n / N}$$

DFT:

$$H_n \equiv \sum_{k=0}^{N-1} h_k \, e^{2\pi i k n / N}$$

IDFT:

$$h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n \, e^{-2\pi i k n / N}$$

Fast Fourier Transform(FFT)

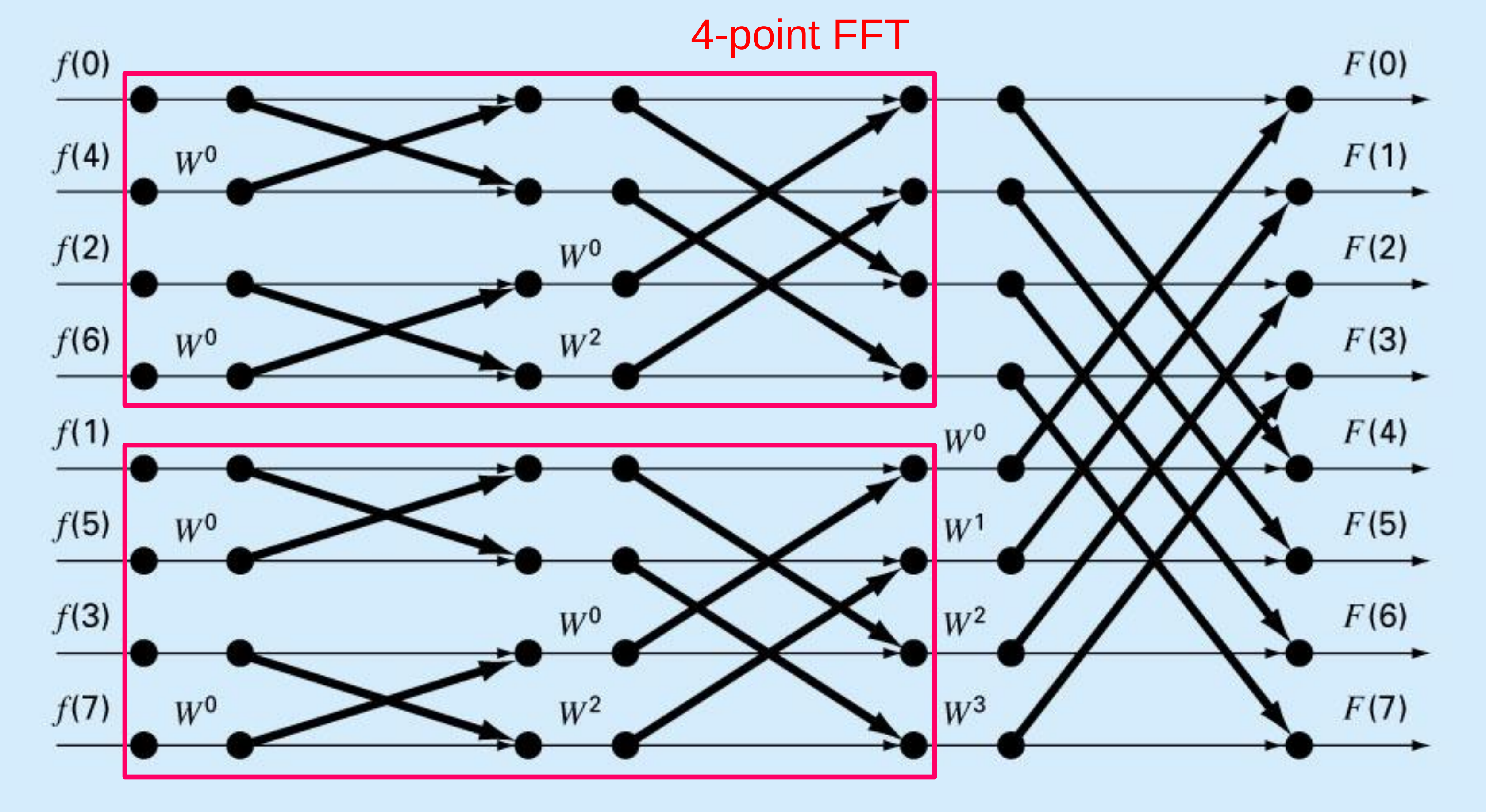
$$H_n = \sum_{k=0}^{N-1} W^{nk} h_k \qquad W \equiv e^{2\pi i/N}$$

[Danielson&Lanczos][Cooley&Tukey]

$$\begin{aligned}
F_k &= \sum_{j=0}^{N-1} e^{2\pi ijk/N} f_j \\
&= \overset{\boxed{\text{even}}}{\sum_{j=0}^{N/2-1} e^{2\pi ik(2j)/N} f_{2j}} + \overset{\boxed{\text{odd}}}{\sum_{j=0}^{N/2-1} e^{2\pi ik(2j+1)/N} f_{2j+1}} \\
&= \sum_{j=0}^{N/2-1} e^{2\pi ikj/(N/2)} f_{2j} + W^k \sum_{j=0}^{N/2-1} e^{2\pi ikj/(N/2)} f_{2j+1} \\
&= F_k^e + W^k\, F_k^o
\end{aligned}$$

Decimation-in-time FFT

Cooley-Tukey Algorithm



Sande-Tukey Algorithm

Power of 2 Sampling

$$N = 2^M$$

Exponential Power Formulation

$$\begin{aligned} F_j &= \sum_{k=0}^{N-1} f_k e^{-i2\pi jk/N}\ , \quad k = 0, 1, \ldots N-1 \\ &= \sum_{k=0}^{N-1} f_k w^{jk}\ , \quad k = 0, 1, \ldots N-1 \end{aligned}$$

$$w = e^{-i2\pi/N}$$

Transform Splitting

$$\begin{aligned} F_j &= \sum_{k=0}^{N/2-1} f_k e^{-i2\pi jk/N} + \sum_{k=N/2}^{N-1} f_k e^{-i2\pi jk/N} \\ &= \sum_{k=0}^{N/2-1} f_k e^{-i2\pi jk/N} + \sum_{m=0}^{N/2-1} f_{m+N/2} e^{-i2\pi j(m+N/2)/N} \\ &= \sum_{k=0}^{N/2-1} (f_k + e^{-i\pi j} f_{k+N/2}) e^{-i2\pi jk/N} \end{aligned}$$

$$e^{-i\pi j} = (-1)^j$$

Even Frequency Numbers

$$\begin{aligned} F_{2j} &= \sum_{k=0}^{N/2-1} (f_k + f_{k+N/2}) e^{-i2\pi 2jk/N} \\ &= \sum_{k=0}^{N/2-1} (f_k + f_{k+N/2}) e^{-i2\pi jk/(N/2)} \\ &= \sum_{k=0}^{N/2-1} (f_k + f_{k+N/2}) w^{2jk} \end{aligned}$$

Odd Frequency Numbers

$$\begin{aligned} F_{2j+1} &= \sum_{k=0}^{N/2-1} (f_k - f_{k+N/2}) e^{-i2\pi(2j+1)k/N} \\ &= \sum_{k=0}^{N/2-1} (f_k - f_{k+N/2}) e^{-i2\pi k/N} e^{-i2\pi jk/(N/2)} \\ &= \sum_{k=0}^{N/2-1} (f_k - f_{k+N/2}) w^k w^{2jk} \end{aligned}$$

Half-size Sequences

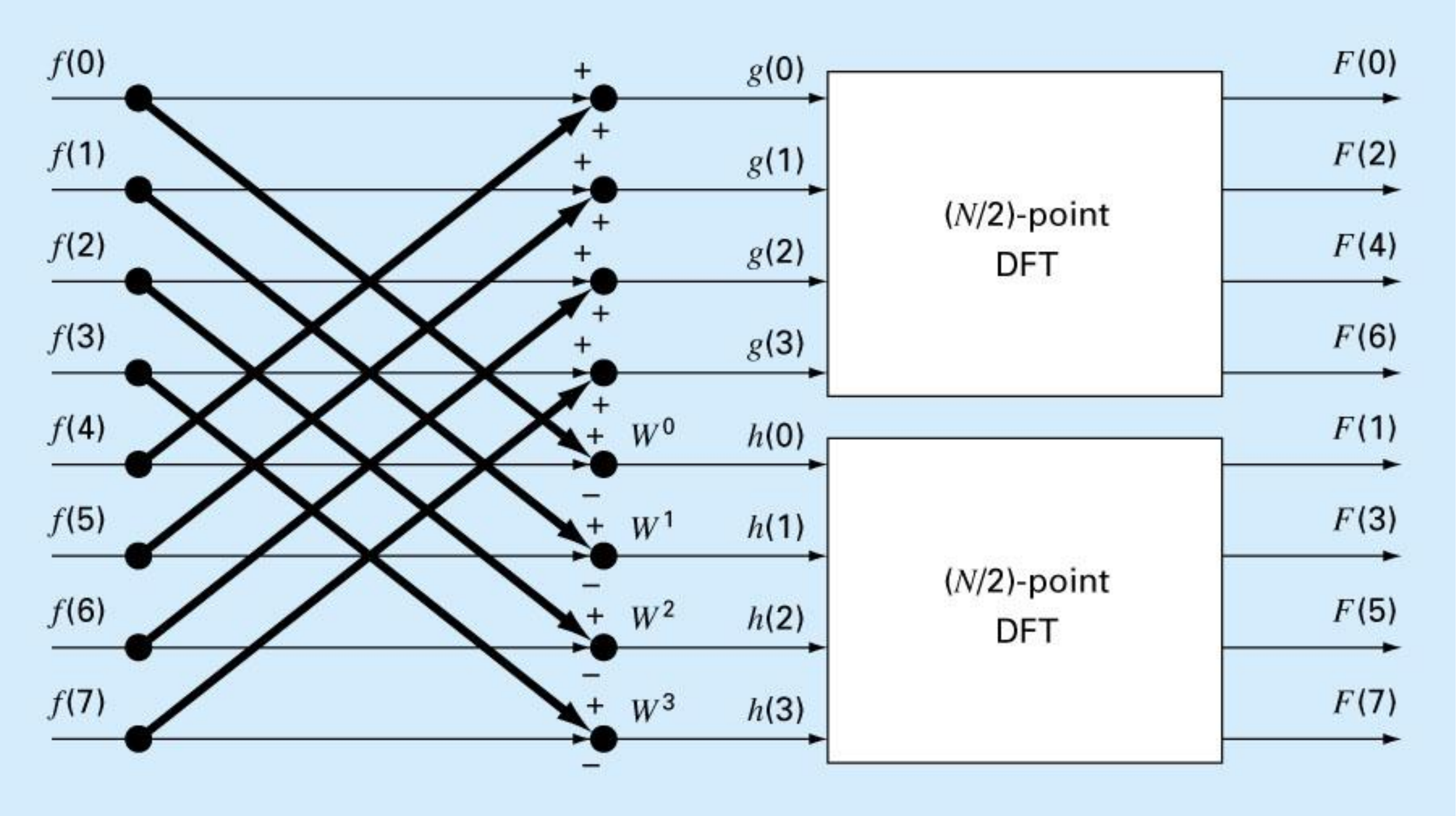
$$\begin{aligned} g_k &= f_k + f_{k+N/2} \\ h_k &= (f_k - f_{k+N/2}) w^k \end{aligned}$$

Half-size Transforms

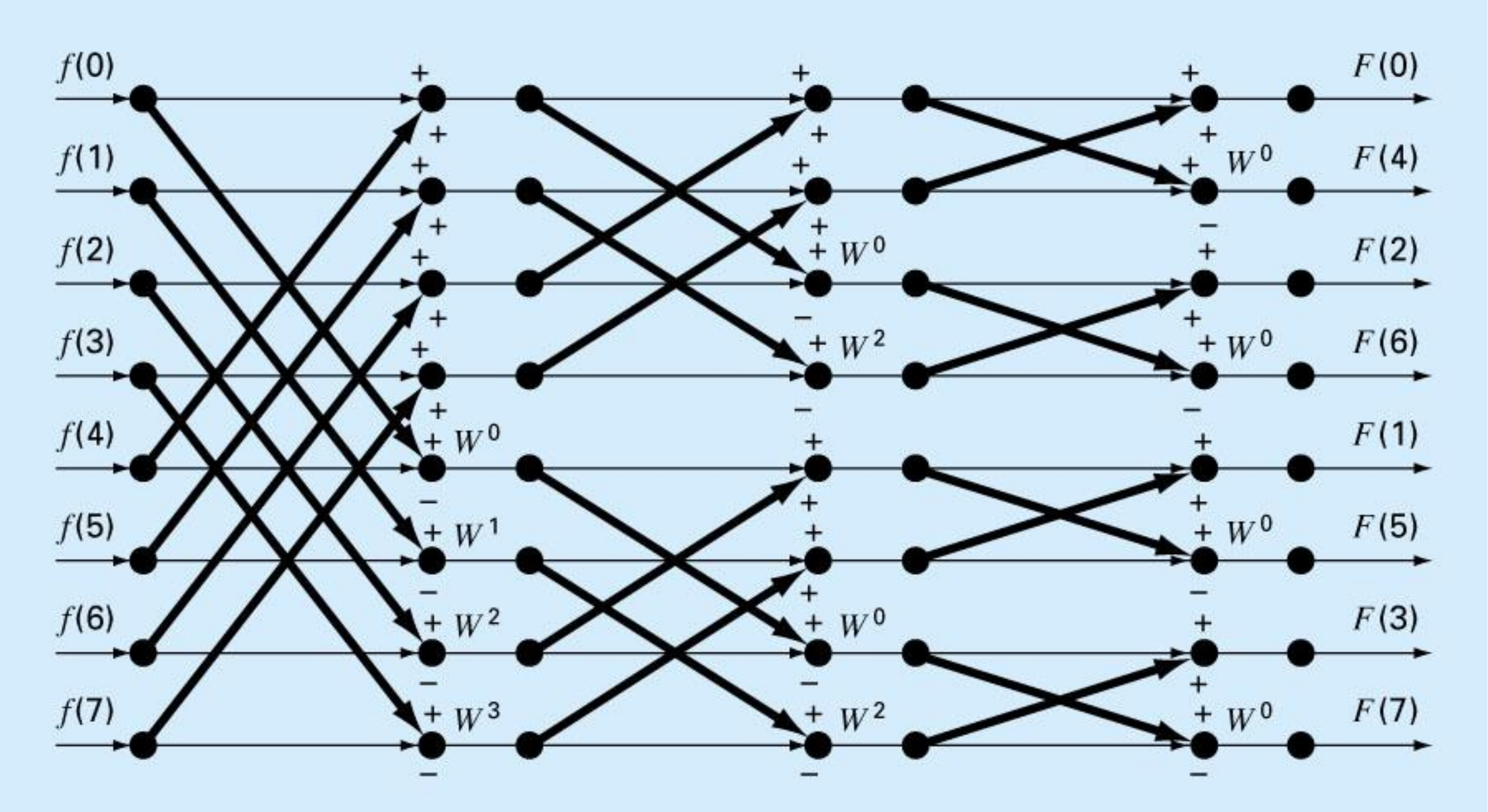
$$\begin{aligned} F_{2j} &= G_j \\ F_{2j+1} &= H_j \end{aligned}$$

Decimation-in-frequency FFT(I)

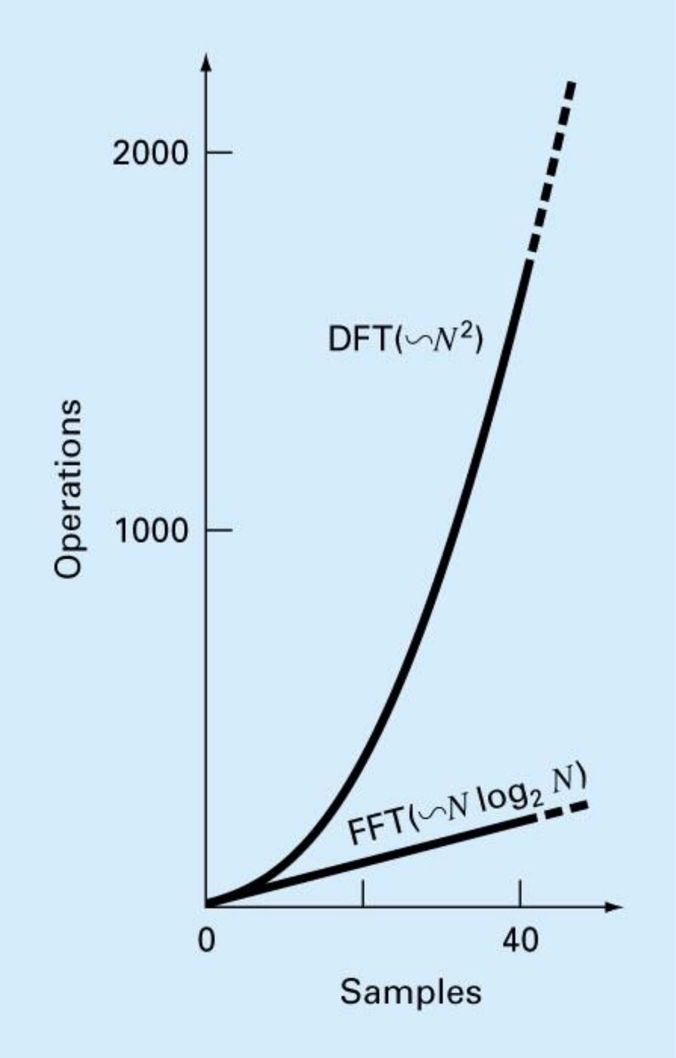
Sande-Tukey Algorithm



Decimation-in-frequency FFT (II)

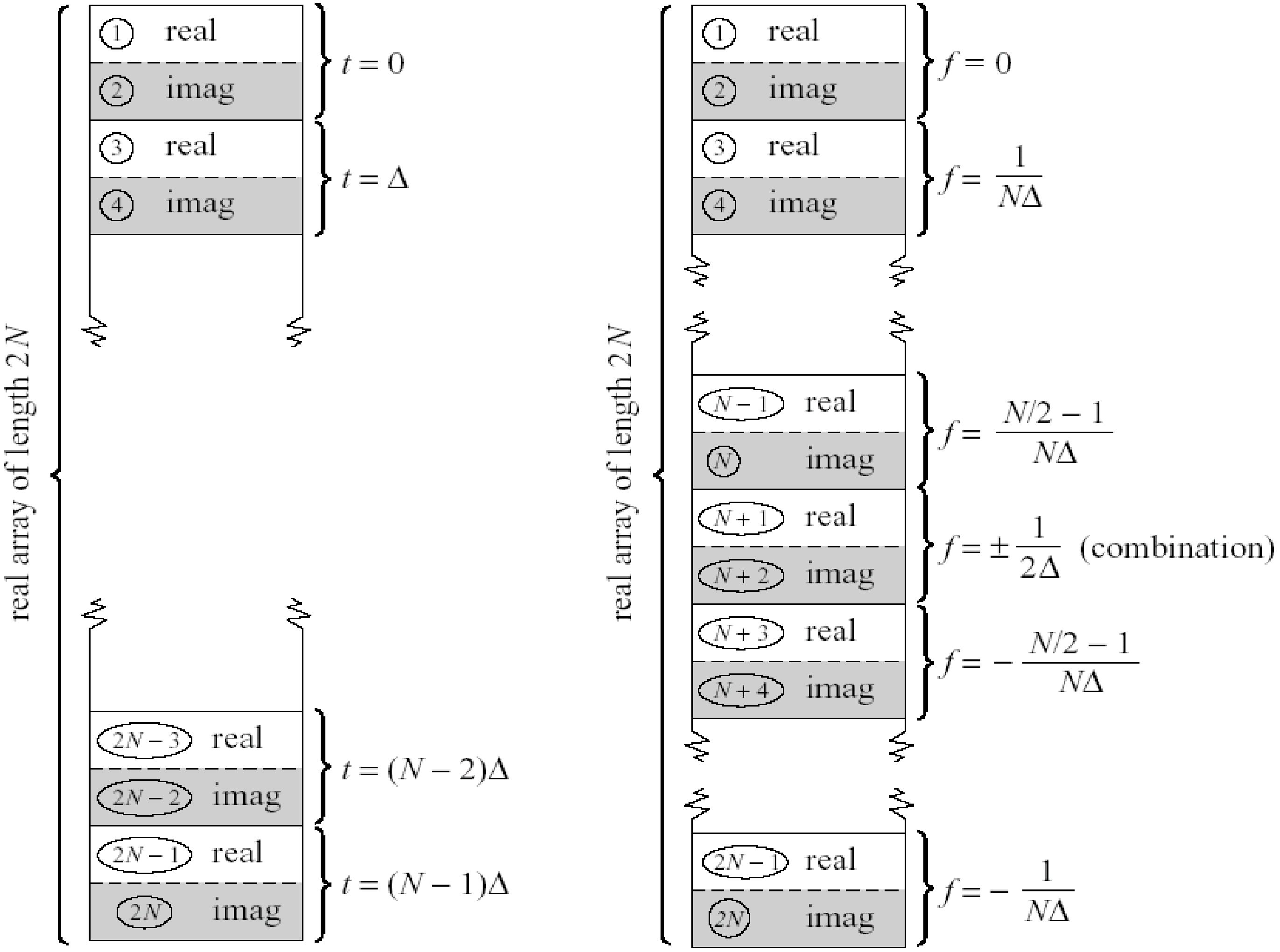


Why FFT?



Further reading: http://en.wikipedia.org/wiki/Cooley-Tukey_FFT_algorithm

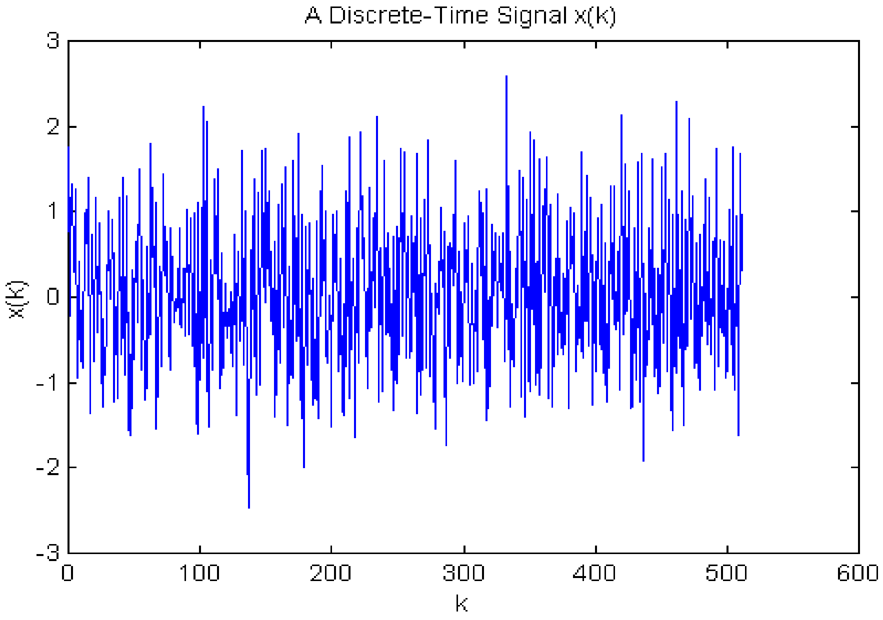
Computation of FFT(I)

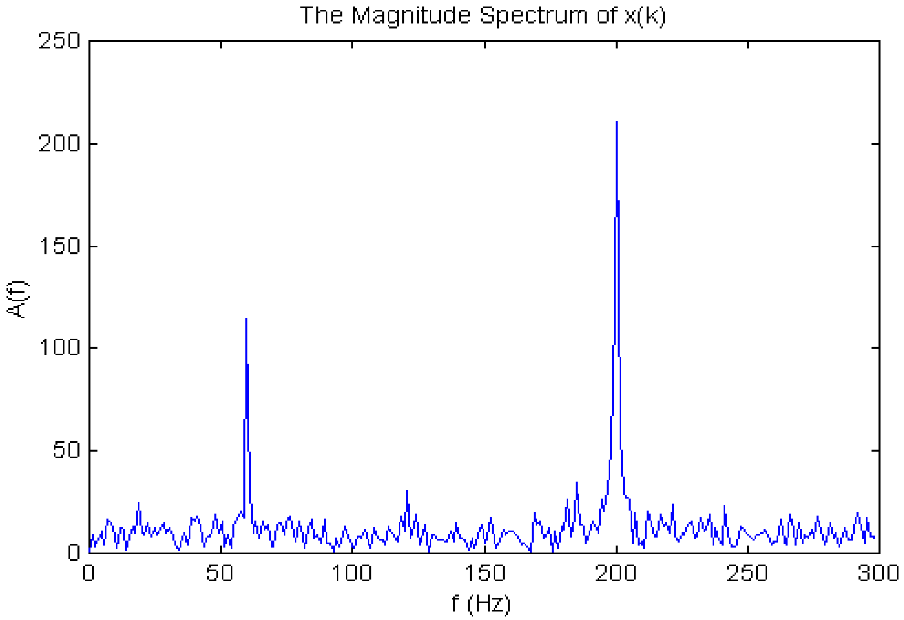


input and output of four1() in NR in C

Computation of FFT(II)

- Eg. FFT

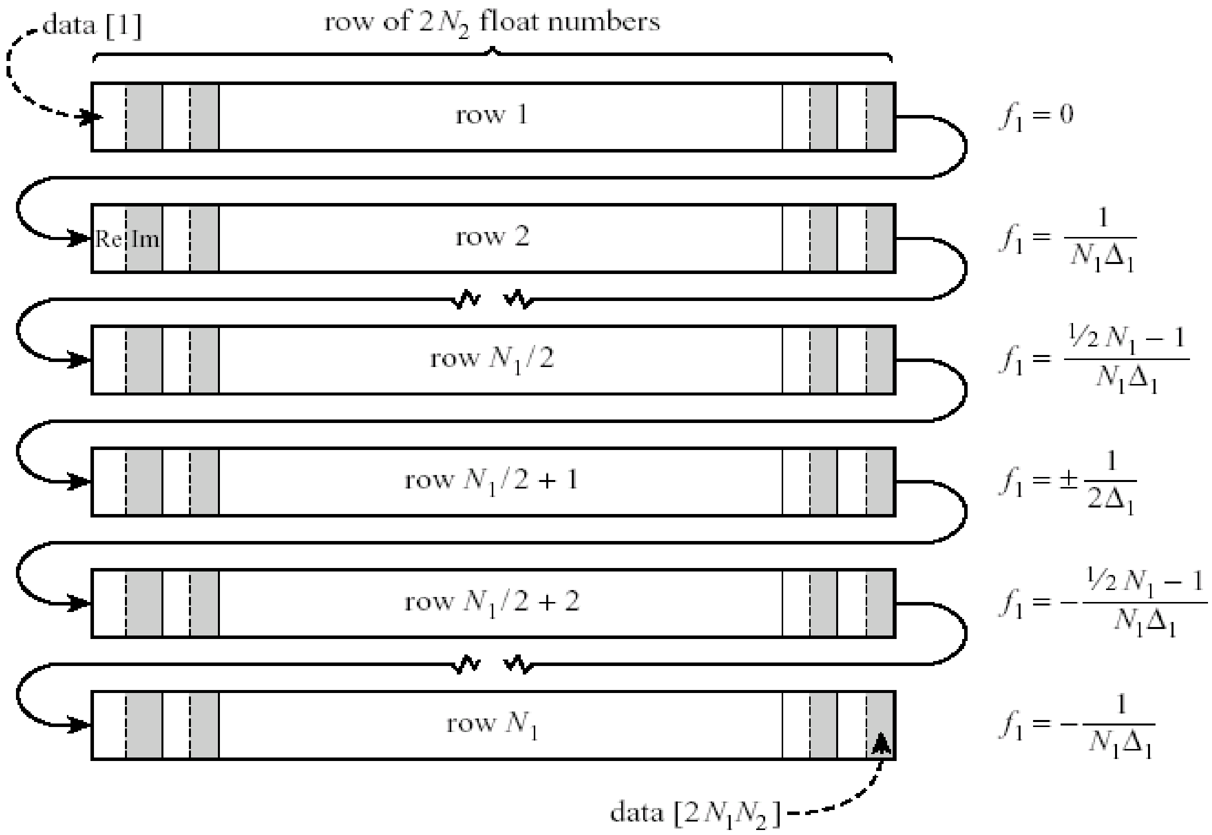




2D FFT(I)

$$H(n_1,n_2) \equiv \sum_{k_2=0}^{N_2-1}\sum_{k_1=0}^{N_1-1} \exp(2\pi i k_2 n_2/N_2)\ \exp(2\pi i k_1 n_1/N_1)\ h(k_1,k_2)$$

$$\begin{aligned} H(n_1,n_2) &= \text{FFT-on-index-1}\,(\text{FFT-on-index-2}\,[h(k_1,k_2)]) \\ &= \text{FFT-on-index-2}\,(\text{FFT-on-index-1}\,[h(k_1,k_2)]) \end{aligned}$$



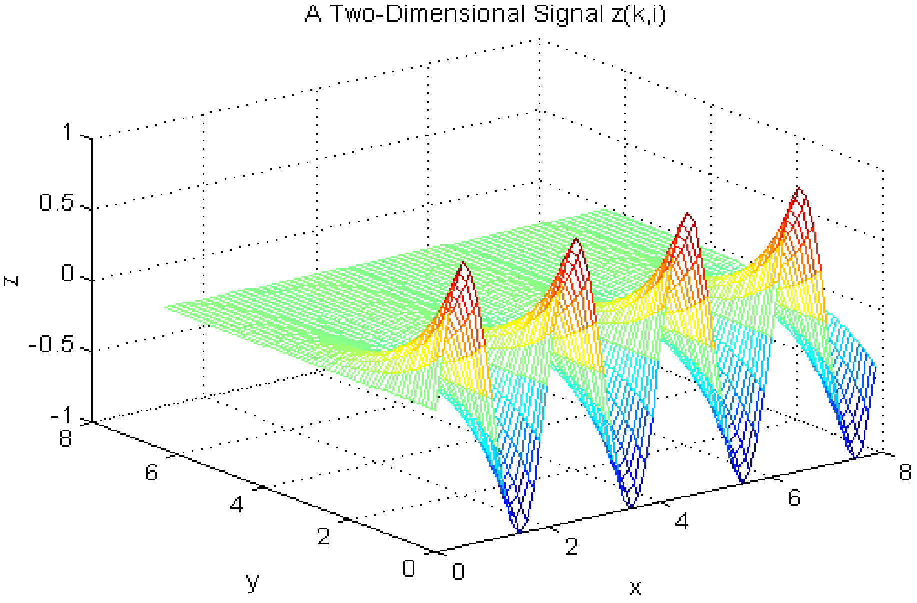
2D FFT(II)

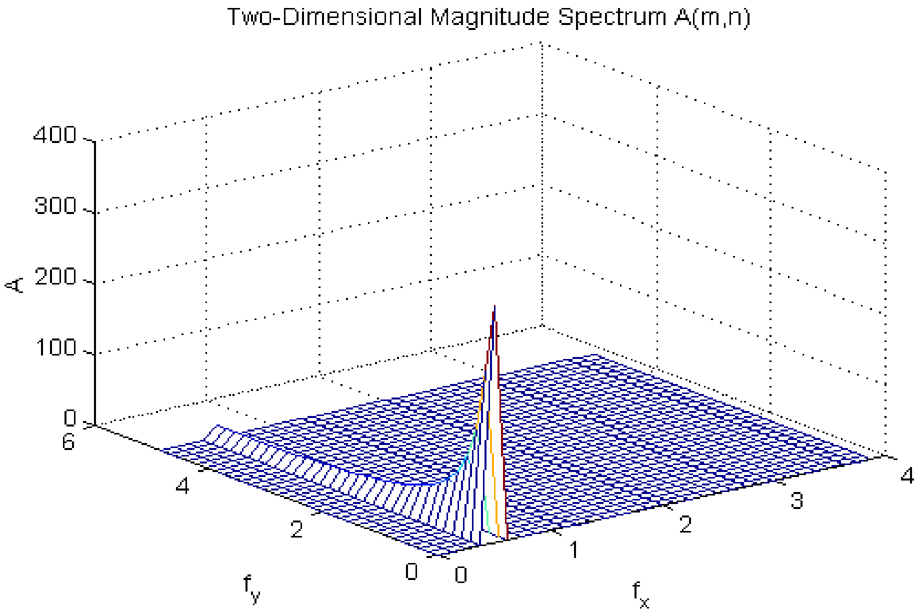
* Generalization to L-dimension

$$H(n_1, \ldots, n_L) \equiv \sum_{k_L=0}^{N_L-1} \cdots \sum_{k_1=0}^{N_1-1} \exp(2\pi i k_L n_L / N_L) \times \cdots \times \exp(2\pi i k_1 n_1 / N_1)\, h(k_1, \ldots, k_L)$$

2D FFT(III)

- Eg. 2D FFT


A Two-Dimensional Signal z(k,i)


Two-Dimensional Magnitude Spectrum A(m,n)