

Quicksort

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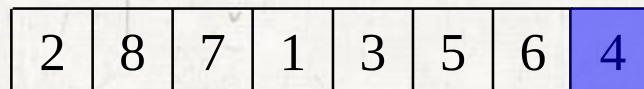
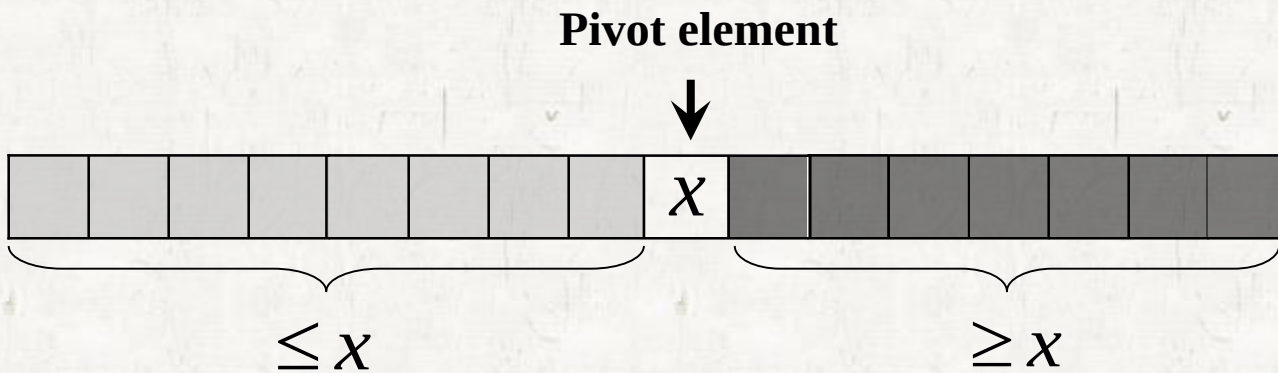
Quicksort

• Divide-and-Conquer paradigm

```
QUICKSORT( $A, p, r$ )  
  if  $p < r$   
     $q = \text{PARTITION}(A, p, r)$   
    QUICKSORT( $A, p, q - 1$ )  
    QUICKSORT( $A, q + 1, r$ )
```

Quicksort

Partition



Quicksort

2	8	7	1	3	5	6	4
---	---	---	---	---	---	---	---

2	1	7	8	3	5	6	4
---	---	---	---	---	---	---	---

2	8	7	1	3	5	6	4
---	---	---	---	---	---	---	---

2	1	3	8	7	5	6	4
---	---	---	---	---	---	---	---

2	8	7	1	3	5	6	4
---	---	---	---	---	---	---	---

2	1	3	8	7	5	6	4
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2	8	7	1	3	5	6	4
---	---	---	---	---	---	---	---

2	1	3	8	7	5	6	4
---	---	---	---	---	---	---	---

2	1	3	4	7	5	6	8
---	---	---	---	---	---	---	---

Quicksort

Partition

```
PARTITION( $A, p, r$ )  
1  $x = A[r]$   
2  $i = p - 1$   
3 for  $j = p$  to  $r - 1$   
4   if  $A[j] \leq x$   
5      $i = i + 1$   
6     exchange  $A[i]$  with  $A[j]$   
7 exchange  $A[i+1]$  with  $A[r]$   
8 return  $i+1$ 
```

Quicksort

- **Partition**

- $\Theta(n)$ time.

- ***Balanced partitioning vs. unbalanced partitioning***

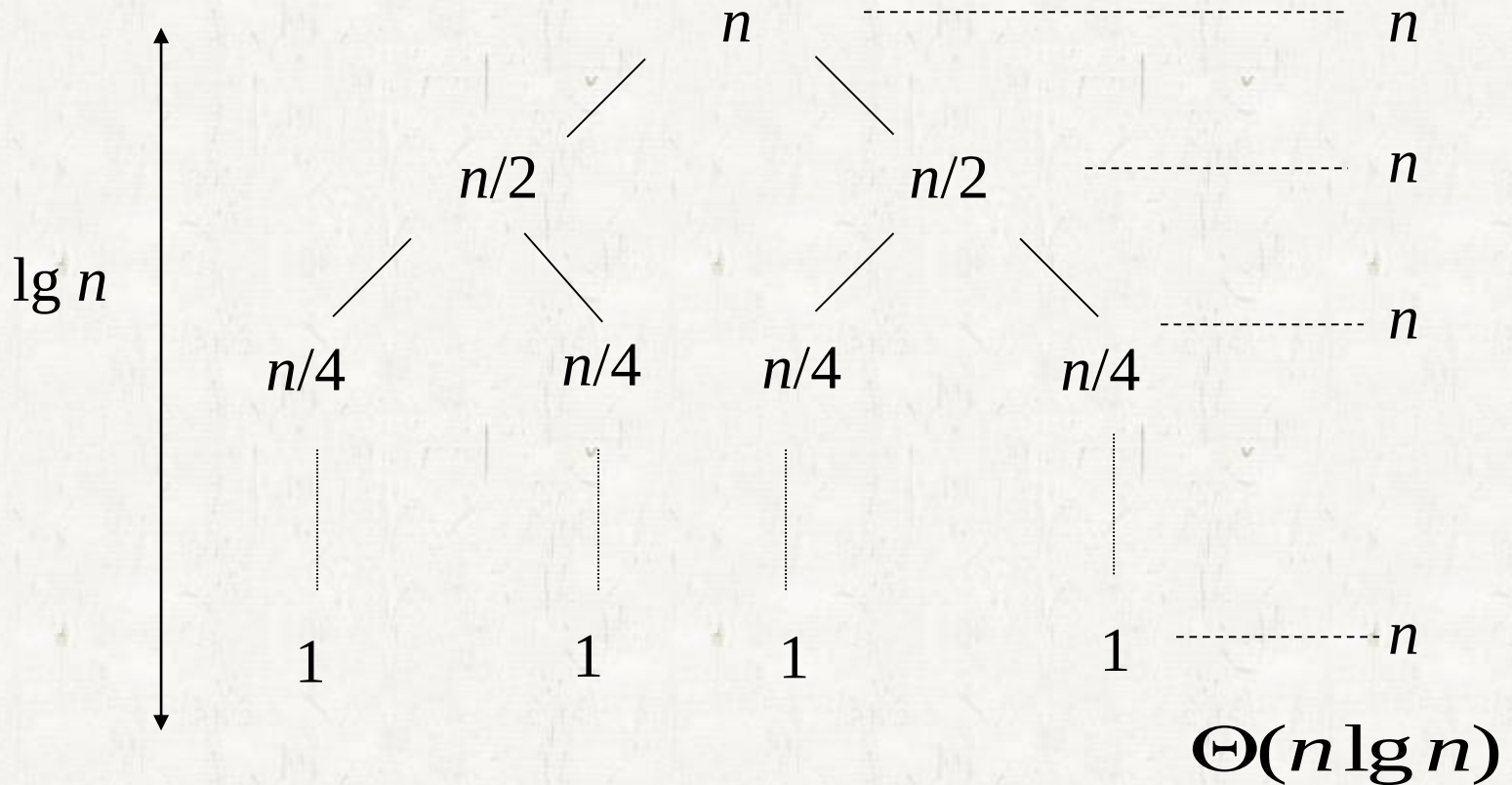
Performance of quicksort

• **Balanced partitioning**

- When PARTITION produces two subproblems of sizes $\lfloor n/2 \rfloor$ and $\lfloor n/2 \rfloor - 1$.
- $T(n) \leq 2T(n/2) + \Theta(n) = O(n \lg n)$

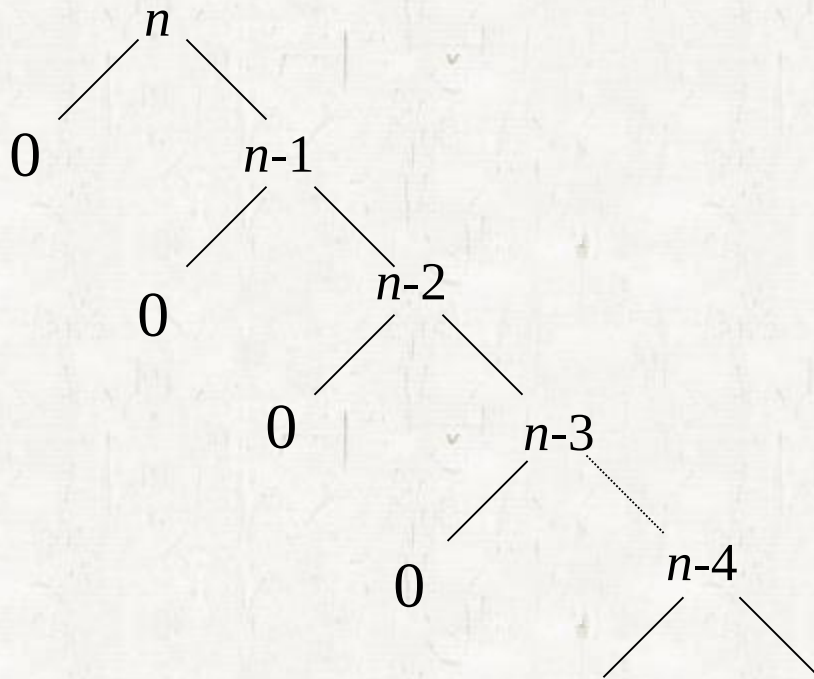
Performance of quicksort

• Balanced partitioning



Performance of quicksort

• Unbalanced partitioning



Performance of quicksort

• Unbalanced partitioning

$$T(n) = T(n-1) + \Theta(n)$$

$$= \sum_{k=1}^n \Theta(k)$$

$$= \Theta\left(\sum_{k=1}^n k\right)$$

$$= \Theta(n^2).$$

Worst-case Analysis

• Worst-case analysis

- Quicksort takes $\Omega(n^2)$ time in worst case.
 - Consider the unbalanced partitioning.
- Is the unbalanced partitioning the worst case?

Worst-case Analysis

Worst-case analysis

- Show that the running time of quicksort is $O(n^2)$ by substitution method.

$$T(n) = \max_{0 \leq q \leq n-1} (T(q) + T(n-q-1)) + \Theta(n)$$

- Show that $T(n) \leq cn^2$ for some constant c .

$$\begin{aligned} T(n) &\leq \max_{0 \leq q \leq n-1} (cq^2 + c(n-q-1)^2) + \Theta(n) \\ &= c \cdot \max_{0 \leq q \leq n-1} (q^2 + (n-q-1)^2) + \Theta(n) \\ &= c \cdot \max_{0 \leq q \leq n-1} (2q^2 - 2q(n-1) + (n-1)^2) + \Theta(n) \\ &= c \cdot \max_{0 \leq q \leq n-1} (2(q - (n-1)/2)^2 + (n-1)^2 / 2) + \Theta(n) \end{aligned}$$

Worst-case Analysis

Worst-case analysis

- The internal expression is maximized when $q = 0$ or $n-1$.

$$\begin{aligned}T(n) &\leq c \cdot \max_{0 \leq q \leq n-1} (2(q - (n-1)/2)^2 + (n-1)^2 / 2) + \Theta(n) \\&= c \cdot (n-1)^2 + \Theta(n) \\&= cn^2 - c(2n-1) + \Theta(n) \\&\leq cn^2\end{aligned}$$

- We can pick the constant c large enough so that the $c(2n-1)$ term dominates the $\Theta(n)$ term.
- Thus, $T(n) = O(n^2)$.

Average-case Analysis

• Average-case analysis

$$\begin{aligned} E[T(n)] &= \frac{1}{n} \left(\sum_{q=1}^n (E[T(q-1)] + E[T(n-q)]) + \Theta(n) \right) \\ &= \frac{2}{n} \left(\sum_{q=2}^{n-1} (E[T(q)]) \right) + \Theta(n) \end{aligned}$$

- By substitution method, show $T(n) \leq cn \lg n$ for some c .
- Problem 7-3.

Average-case Analysis II

• Average Case Analysis II

- Let X be the number of comparisons over the entire execution of QUICKSORT on an n -element array.
- Then the average running time of QUICKSORT is
 - $O(n + E[X])$.
- We will not attempt to analyze how many comparisons are made in *each* PARTITION.
- Rather, we will derive an overall bound on the total number of comparisons.

Average-case Analysis II

- Let z_i denote the i th smallest element in the sorted array.
- $Z_{ij} = \{z_i, z_{i+1}, \dots, z_j\}$
- Each pair of elements z_i and z_j is compared at most once.
 - An element is compared only to the pivot element in each PARTITION.
 - The pivot element used in a PARTITION is never again compared to any other elements.

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^n \Pr\{z_i \text{ is compared to } z_j\}$$

Average-case Analysis II

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^n \Pr\{z_i \text{ is compared to } z_j\}$$

- $\Pr\{z_i \text{ is compared to } z_j\}$
 - $\Pr\{z_i \text{ or } z_j \text{ is first pivot chosen from } Z_{ij}\}$

$$= \frac{2}{j-i+1}$$

$$E[x] = \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{2}{j-i+1}$$

Average-case Analysis II

$k = j - i$, the harmonic series

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{2}{j-i+1}$$

$$= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1}$$

$$< \sum_{i=1}^{n-1} \sum_{k=1}^n \frac{2}{k}$$

$$= \sum_{i=1}^{n-1} O(\lg n)$$

equation(A.7)

$$= O(n \lg n)$$

Randomized quicksort

RANDOMIZED-PARTITION(A, p, r)

1. $i = \text{RANDOM}(p, r)$
2. exchange $A[r]$ with $A[i]$
3. **return** **PARTITION**(A, p, r)

Randomized quicksort

RANDOMIZED-QUICKSORT(A, p, r)

1 **if** $p < r$

2 $q = \text{RANDOMIZED-PARTITION}(A, p, r)$

3 RANDOMIZED-QUICKSORT($A, p, q - 1$)

4 RANDOMIZED-QUICKSORT($A, q + 1, r$)

Self-study

- **Exercise 7.1-2**
 - Balanced partition with same elements
- **Exercise 7.2-4**
 - Sorting almost-sorted input